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Northwestern University, Methods for Theroetical Physicists
Mathematical Methods for Physicists, Seventh Ed.- Arfken
November 2, 2015

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1 Arfken 11.8.10.

Show that

$$\int_0^\infty \frac{x \sin x}{x^2+1} dx = \frac{\pi}{2e} \ . \tag{1}$$

First define $f(x)$ as the integrand, and replace the $\sin x$ term with the complex exponential representation,

$$f(x) = \frac{x\frac{1}{2i}(e^{ix} - e^{-ix})}{x^2+1} = \frac{1}{2i}\left[\frac{xe^{ix}}{x^2+1} - \frac{xe^{-ix}}{x^2+1}\right], \tag{2}$$

which makes the integral of interest,

$$I \equiv \int_0^\infty f(x)dx = \frac{1}{2i}\left[\int_0^\infty \frac{xe^{ix}}{x^2+1}dx - \int_0^\infty \frac{xe^{-ix}}{x^2+1}dx\right] = \frac{1}{2i}\left[\int_0^\infty \frac{xe^{ix}}{x^2+1}dx + \int_\infty^0 \frac{xe^{-ix}}{x^2+1}dx\right] \ , \tag{3}$$

by flipping the limits of integration and obtaining a factor of -1 for the second integral. By substituting $x \to -x$ in the same integral, the limits of integration will span the entire real line. This causes the integrand to pick up two factors of -1 (one from x and one from dx), and $(-x)^2 = x^2$, therefore the only change to the integrand is $e^{-ix} \to e^{ix}$. This substitution makes the integrands equal, and the limits of integration continuous, allowing the integral to be rewritten as

$$I = \frac{1}{2i}\int_{-\infty}^\infty \frac{xe^{ix}}{x^2+1}dx \ , \tag{4}$$

which can be solved using a contour integral, with one section spanning the entire real axis. Rewriting this integral with a complex-valued ($x \to z$) function as a contour integral gives

$$\oint_C \phi(z)dz = \oint_C \frac{ze^{iz}}{z^2+1}dz = 2\pi i \sum_n \underset{z\to z_n}{Res}\ \phi(z) \ , \tag{5}$$

where z_n is the n^{th} pole of $\phi(z)$. Note that $\phi(z)$ has poles at $z_n = \pm i$. It is clear that in the upper half-plane, $\phi(z)$ is analytic except for the pole at $z_0 = i$. Choosing C such that it is a closed semi-circle in the upper half-plane of infinite radius, allows the integral to be written,

$$\oint_C \phi(z)dz = 2\pi i \underset{z\to i}{Res}\ \phi(z) = \lim_{R\to\infty}\left[\int_{-R}^R \phi(z)dz + \int_{C_R}\phi(z)dz\right] = 2iI + \lim_{R\to\infty}\int_{C_R}\phi(z)dz \ , \tag{6}$$

where C_R is the circular arc in the upper half-plane, going from $\theta = 0$ to $\theta = \pi$. The integral along the circular arc goes to zero because as $|z| \to \infty$ in the upper half plane, the exponential is negligible, so $\phi(z)$ dies slightly faster than $1/z$ (see Arfken page 525). This gives the value of the desired integral to be

$$I = \pi \underset{z\to i}{Res}\ \phi(z) \ . \tag{7}$$

The value of the residue is calculated in the standard way,

$$\underset{z\to i}{Res}\ \phi(z) = \lim_{z\to i}(z-i)\frac{ze^{iz}}{(z+i)(z-i)} = \frac{ie^{-1}}{2i} = \frac{1}{2e} \ , \tag{8}$$

so that

$$\int_0^\infty \frac{x \sin x}{x^2+1} dx = \frac{\pi}{2e} \ . \tag{9}$$

2 Arfken 11.8.17.

Show that

$$I \equiv \int_0^\infty \frac{x^p \ln x}{x^2+1} dx = \frac{\pi^2}{4} \frac{\sin(\pi p/2)}{\cos^2(\pi p/2)} \ , \tag{10}$$

for $0 < p < 1$. As in the previous problem, this will eventually be evaluated using a contour integral; begin by defining a function

$$f(z) = \frac{z^p \ \mathrm{Ln}(z)}{z^2+1} \ , \tag{11}$$

so that

$$\oint_C f(z) dz = 2\pi i \sum_n \underset{z \to z_n}{Res} f(z) \ , \tag{12}$$

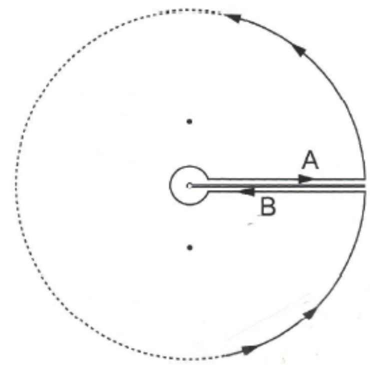

Figure 1: Contour used in evaluating the integral in Arfken 11.8.17.

where $\mathrm{Ln}(z)$ is the complex logarithm. Inspection of this function shows there are two simple poles at $z_n = \pm i$. Expanding the complex logarithm for $z = re^{i\theta}$, gives $\mathrm{Ln}(z) = \ln(r) + i\theta$, which says there exists a branch point at $z = 0$. Therefore a branch cut is introduced along the positive real axis. Following Arfken Example 11.8.8, the contour C is defined to be a line just above the positive real axis (where $z^p = x^p$ and $\mathrm{Ln}(z) = \ln x$) integrated to the right, which is connected to a circle of infinite radius that terminates at a line just below the positive real axis. The final arc of the contour is a circle of infinitesimal radius ϵ around the origin connecting the two parallel segments (see Arfken Figure 11.26). This makes the contour integral

$$\oint_C f(z) dz = \lim_{R \to \infty} \lim_{\epsilon \to 0} \left[\int_{C_R} f(z) dz + \int_{C_\epsilon} f(z) dz + \int_\epsilon^R f(z) dz + \int_R^\epsilon f(z) dz \right] \ , \tag{13}$$

where the first line integral (ϵ to R) is the integral of interest, define this segment as A, and the other linear path as B. As in the example, both integrals over circular arcs do not contribute to this sum, for $f(z)$ dies as $\sim 1/z$. Along path B, $z = re^{2\pi i}$, therefore its contribution is

$$\lim_{R \to \infty} \lim_{\epsilon \to 0} \int_R^\epsilon f(z) dz = \int_\infty^0 \frac{r^p e^{2\pi i p} [\ln(r) + \ln(e^{2\pi i})]}{r^2 e^{4\pi i} + 1} dr \tag{14}$$

$$= - \left[\int_0^\infty \frac{r^p e^{2\pi i p} \ln(r)}{r^2+1} dr + \int_0^\infty \frac{r^p e^{2\pi i p}(2\pi i)}{r^2+1} dr \right] \ , \tag{15}$$

by noting that $e^{n\pi i} = 1$ for even n. Along path A, $z = r$ because $\theta = 0$, so the integral of interest is

$$I = \lim_{R \to \infty} \lim_{\epsilon \to 0} \int_\epsilon^R f(z) dz = \int_0^\infty \frac{r^p \ln(r)}{r^2+1} dr \ , \tag{16}$$

from this the integral along path B becomes

$$- \left[e^{2\pi i p} \int_0^\infty \frac{r^p \ln(r)}{r^2+1} dr + 2\pi i e^{2\pi i p} \int_0^\infty \frac{r^p}{r^2+1} dr \right] = - \left[e^{2\pi i p} I + 2\pi i e^{2\pi i p} \frac{\pi}{2\cos(p\pi/2)} \right] \ , \tag{17}$$

using the result of Arfken Example 11.8.8. Therefore, using Equations 11 and 13, and the values for the integrals,

$$2\pi i \left[\underset{z \to i}{Res} f(z) + \underset{z \to -i}{Res} f(z) \right] = I - e^{2\pi i p} I - 2\pi i e^{2\pi i p} \frac{\pi}{2\cos(p\pi/2)} \ , \tag{18}$$

which yields,

$$I = \frac{1}{1 - e^{2\pi ip}} \left[2\pi i(B_+ + B_-) + i\frac{\pi^2 e^{2\pi ip}}{\cos(p\pi/2)} \right] , \tag{19}$$

where B_+ and B_- are the residues for $+i$ and $-i$, respectively. The only remaining step is to calculate the residues: begin by writing the poles in polar form, $z_+ = e^{i\pi/2}$ and $z_- = e^{i3\pi/2}$. The residues can now be calculated in the standard way

$$B_i = \lim_{z \to z_i} (z - z_i) \frac{z^p \, \text{Ln}(z)}{(z + z_i)(z - z_i)} = \frac{z_i^p \ln(z_i)}{2z_i} , \tag{20}$$

yeilding the results

$$B_+ = e^{i\pi p/2} \frac{i\pi/2}{2i} \quad \text{and} \quad B_- = e^{3i\pi p/2} \frac{3i\pi/2}{-2i} \tag{21}$$

$$B_+ + B_- = \frac{\pi}{4}(e^{i\pi p/2} - 3e^{3i\pi p/2}) , \tag{22}$$

the exponentials are not simplified to $\pm i$ so they can be substituted for trig functions later. This gives the value for the integral of interest

$$(1 - e^{2\pi ip})I = \left[2\pi i \frac{\pi}{4}(e^{i\pi p/2} - 3e^{3i\pi p/2}) + i\frac{\pi^2 e^{2\pi ip}}{\cos(p\pi/2)} \right] \tag{23}$$

$$= \frac{i\pi^2}{2} \left[e^{i\pi p/2} - 3e^{3i\pi p/2} + \frac{2e^{2\pi ip}}{\cos(p\pi/2)} \right] . \tag{24}$$

To remove the factors of 2 in the exponentials, both sides are multiplied by a factor of $e^{-i\pi p}$, which yields,

$$(e^{-i\pi p} - e^{i\pi p})I = \frac{i\pi^2}{2} \left[e^{-i\pi p/2} - 3e^{i\pi p/2} + \frac{2e^{\pi ip}}{\cos(p\pi/2)} \right] . \tag{25}$$

Though they were used previously, it is handy to note the Euler identities

$$\begin{cases} \sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}) \\ \cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}) \end{cases} \quad \text{and} \quad \begin{cases} e^{ix} = \cos x + i\sin x \\ e^{-ix} = \cos x - i\sin x \end{cases} , \tag{26}$$

which when employed, Equation 25 becomes

$$-2i\sin(p\pi)I = \frac{i\pi^2}{2} \left[e^{-i\pi p/2} - 3e^{i\pi p/2} + \frac{2e^{\pi ip}}{\cos(p\pi/2)} \right] \tag{27}$$

$$\sin(p\pi)I = \frac{\pi^2}{4} \left[3e^{i\pi p/2} - e^{-i\pi p/2} - \frac{2e^{\pi ip}}{\cos(p\pi/2)} \right] \tag{28}$$

$$= \frac{\pi^2}{4} \left[2e^{i\pi p/2} + e^{i\pi p/2} - e^{-i\pi p/2} - \frac{2(e^{\pi ip/2})^2}{\cos(p\pi/2)} \right] \tag{29}$$

$$= \frac{\pi^2}{4} \left[2e^{i\pi p/2} + 2i\sin(p\pi/2) - \frac{2[\cos(p\pi/2) + i\sin(p\pi/2)]^2}{\cos(p\pi/2)} \right] \tag{30}$$

$$= \frac{\pi^2}{4} \left[2\cos(\pi p/2) + 2i\sin(\pi p/2) + 2i\sin(p\pi/2) - \frac{2[\cos(p\pi/2) + i\sin(p\pi/2)]^2}{\cos(p\pi/2)} \right] \tag{31}$$

$$= \frac{\pi^2}{4} \left[2C(\pi p/2) + 4iS(\pi p/2) - 2\frac{[C^2(p\pi/2) - S^2(p\pi/2) + 2iC(p\pi/2)S(p\pi/2)]}{C(p\pi/2)} \right] , \tag{32}$$

where $S(x) = \sin(x)$ and $C(x) = \cos(x)$. Making the transformation that $\sin(p\pi) = 2\cos(p\pi/2)\sin(p\pi/2)$, yields

$$2IC(p\pi/2)S(p\pi/2) = \frac{\pi^2}{4}\left[2\frac{S^2(p\pi/2)}{C(p\pi/2)}\right] \quad \Rightarrow \quad I = \frac{\pi^2}{4}\left[\frac{\sin(p\pi/2)}{\cos^2(p\pi/2)}\right] \ . \tag{33}$$

3 Arfken 11.8.18b.

Show that

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2}dx = \frac{\pi^3}{8} \ , \tag{34}$$

by noting that the suggested transformation is $x \to z = e^t$, using the suggested contour in Figure 2 and letting $R \to \infty$. Define the integrand as $f(x)$. Under this transformation, $dx \to e^t dt$ and the limits of integration become $-\infty$ to ∞.

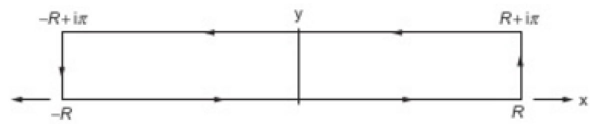

Figure 2: Contour suggested to evaluate the integral in Arfken 11.8.18b.

Using the transformation above the integrand becomes

$$f(x) \to f(z) = \frac{(\ln z)^2}{1+z^2} \to f(t) = \frac{t^2}{1+e^{2t}} \ , \tag{35}$$

therefore the contour integral around the contour in Figure 2 becomes

$$\oint_C f(z)dz = \oint_C f(t)e^t dt = \oint \frac{t^2 e^t}{1+e^{2t}}dt = \oint \frac{t^2}{e^{-t}+e^t}dt = \oint \frac{t^2}{2\cosh(t)}dt \ . \tag{36}$$

Note that under this transformation, the integral of interest is $I = \int_{-\infty}^{\infty}(t^2e^t)/(1+e^{2t})\ dt$, which is equivalent to the expression in Equation 34. Define the integrand (in terms of t) from Equation 36 to be $\phi(t)$,

$$\phi(t) = \frac{t^2 e^t}{1+e^{2t}} = \frac{t^2}{e^{-t}+e^t} = \frac{t^2}{2\cosh(t)} \ , \tag{37}$$

Now the contour integral becomes

$$\oint_C \phi(t)dt = \lim_{R\to\infty}\left[\int_{-R}^{R}\phi(t)dt + \int_{R}^{R+i\pi}\phi(t)dt + \int_{R+i\pi}^{-R+i\pi}\phi(t)dt + \int_{-R+i\pi}^{-R}\phi(t)dt\right] \ , \tag{38}$$

where the first integral is I. The integral along the vertical at $+R$ is

$$\int_R^{R+i\pi} \frac{t^2}{e^{-t}+e^t}dt \ , \tag{39}$$

which by examination of the real part, vanishes as t^2/e^t as $|t| \to \infty$, which is faster than $1/t$, so this integral does not contribute to the contour. The same argument can be used for the vertical segment at $-R$. The final part of the puzzle is the horizontal integral at $+i\pi$. Under a change of variables such that $t' = t + i\pi$ (so $dt' = dt$), this integral becomes

$$\int_{R+i\pi}^{-R+i\pi}\phi(t)d = \int_R^{-R}\phi(t+i\pi)dt = \int_R^{-R}\frac{(t+i\pi)^2e^{t+i\pi}}{1+e^{2(t+i\pi)}}dt = \int_R^{-R}\frac{(t+i\pi)^2e^te^{i\pi}}{1+e^{2t}e^{2i\pi}}dt \tag{40}$$

$$= -\int_{-R}^{R}\frac{-(t+i\pi)^2e^t}{1+e^{2t}}dt = \int_{-R}^{R}\frac{t^2e^t}{1+e^{2t}}dt - \pi^2\int_{-R}^{R}\frac{e^t}{1+e^{2t}}dt + 2i\pi\int_{-R}^{R}\frac{te^t}{1+e^{2t}}dt \ , \tag{41}$$

Note that the first integral is I and the last is zero, because in the limit $R \to \infty$, this becomes an integral of an odd function over all space, which is zero. All that is left in the evaluation of this integral is to evaluate the middle integral,

$$\pi^2\int_{-R}^{R}\frac{e^t}{1+e^{2t}}dt = \frac{\pi^2}{2}\int_{-R}^{R}\frac{1}{\cosh(t)}dt, \tag{42}$$

this is an elementary integral and can be looked up in tables, its value is $\pi/2$.

Using this information, by the Cauchy integral formula, Equation 38 becomes

$$2\pi i \sum_j B_j = I + I - \frac{\pi^3}{2} \ , \tag{43}$$

where B_j is the residue of the j^{th} pole (located at $z_j \to t_j$). The function $f(t)$, given in Equation 35, has poles at $i\pi n/2$, where n is any odd integer. However, by adding a branch cut down the negative imaginary axis, the number of poles can be limited to a finite number. There is a branch point at zero, so by adjusting the contour by adding a semicircular arc of infinitesimal radius around the origin, it can be avoided. This segment does not contribute to the total contour because $f(t) \to 0$ as $|t| \to 0$, so the integral vanishes. This says that there is one pole contained in the contour, $t = i\pi/2$. To find this residue B, L'Hôpital's rule is applied to $f(t)$,

$$B = \lim_{t \to i\pi/2} \frac{t^2(t - \frac{i\pi}{2})}{2\cosh(t)} = \lim_{t \to i\pi/2} \frac{3t^2 - i\pi t}{2\sinh(t)} = \frac{-\frac{3}{4}\pi^2 + \frac{1}{2}\pi^2}{2\sinh\frac{i\pi}{2}} = \frac{-\frac{1}{4}\pi^2}{2i} \ , \tag{44}$$

which gives an expression for the integral of interest, from Equation 43,

$$I = \pi i B + \frac{\pi^3}{4} \quad \Rightarrow \quad I = -\frac{\pi^3}{8} + \frac{\pi^3}{4} \ , \tag{45}$$

which says the integral of interest is

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx = \frac{\pi^3}{8} \ . \tag{46}$$

4 Arfken 11.8.22.

Show that

$$\int_0^\infty \frac{1}{1+x^n}dx = \frac{\pi/n}{\sin(\pi/n)} \ , \qquad (47)$$

using the contour shown in Figure 3, with $\theta = 2\pi/n$. Define the integrand as $f(x)$, and the contour shown in Figure 3 as C. Note that along the path down the positive real axis (path A), $z = x$, so this integral is the integral of interest, I. Along the other linear path (path B), $z = re^{2\pi i/n}$, so that $dz = dr(e^{2\pi i/n})$.

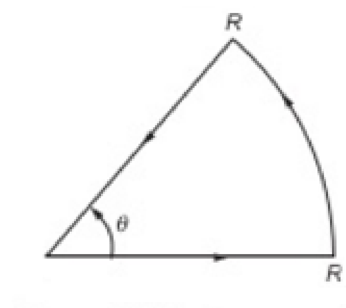

Figure 3: Sector contour used in evaluating the integral in Arfken 11.8.22, with $\theta = 2\pi/n$.

Therefore the integral of $f(z)$, where z is complex, is

$$2\pi i \sum_k \underset{z\to z_k}{Res} f(z) = \oint_C f(z)dz = \lim_{R\to\infty}\left[\int_0^R f(z)dz + \int_{C_R} f(z)dz + \int_R^0 f(z)dz\right] \ , \qquad (48)$$

where z_i are the poles contained in the contour C, and the integral along the circular arc of radius R, C_R vanishes. This is the case because making the assumption that $n > 1$, f(z) dies faster than $1/z$. This function has simple poles at the n^{th} roots of $+1$, denoted by B_k. This makes the expression for the contour integral

$$I - \int_0^\infty f(z)dz = 2\pi i \sum_k B_k \ , \qquad (49)$$

where the limits of integration of the remaining integral were swapped, acquiring a factor of -1. This integral can be written as

$$\int_0^\infty f(z)dz = \int_0^\infty \frac{1}{1+z^n}dz = \int_0^\infty \frac{e^{2\pi i/n}}{1+r^n e^{2n\pi i/n}}dr = e^{2\pi i/n}\int_0^\infty \frac{1}{1+r^n} = e^{2\pi i/n} I \ , \qquad (50)$$

which makes Equation 48 into

$$I(1-e^{2\pi i/n}) = 2\pi i \sum_k B_k \ . \qquad (51)$$

Using the trig identities used in Arfken 11.8.17, and multiplying through by a factor of $e^{-\pi i/n}$, the $(1-e^{2\pi i/n})$ term can be replaced,

$$I(-2i\sin(\pi/n)) = e^{-\pi i/n} 2\pi i \sum_k B_k \quad \Rightarrow \quad I = -\frac{\pi e^{-\pi i/n} B_n}{\sin(\pi/n)} \ . \qquad (52)$$

The sum of all residues can be reduced to a single residue because the angle for path B depends on n. This splits the circle of infinite radius into n evenly sized sectors, each containing exactly one pole, denoted by B_n, located at $z_n = e^{\pi i/n}$. The residue of this pole can be found the standard way,

$$B_n = \lim_{z\to z_n}(z-z_n)\frac{1}{1+z^n} = \lim_{z\to z_n}\frac{1}{nz^{n-1}} = \frac{z_n^{1-n}}{n} = \frac{z_n(z_n^{-n})}{n} = \frac{e^{\pi i/n}}{n}e^{-n\pi i/n} = -\frac{e^{\pi i/n}}{n} \ , \qquad (53)$$

using L'Hôpital's rule. Combining this with Equation 52, gives the final result

$$I = -\frac{\pi e^{-\pi i/n}}{\sin(\pi/n)}\left(-\frac{e^{\pi i/n}}{n}\right) = \frac{\pi/n}{\sin(\pi/n)} = \int_0^\infty \frac{1}{1+x^n}dx \tag{54}$$

5 Arfken 11.9.3.

Evaluate

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots \ . \tag{55}$$

This sum can be defined as

$$\mathcal{S} \equiv \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)^3} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^3} \ . \tag{56}$$

However, according to Arfken Table 11.2, in order to use the prescribed method to evaluate the sum, it must be in the form

$$\sum_{n=-\infty}^{\infty}(-1)^n f\left(n+\frac{1}{2}\right) = \sum_j \underset{z\to z_j}{Res}\left[f(z)\pi\sec(\pi z)\right] \ , \tag{57}$$

so the sum is rewritten as

$$\mathcal{S} = \frac{1}{2^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3} = \frac{1}{2^3}\left[\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3}\right] \ . \tag{58}$$

Taking the summand as $f(n+\frac{1}{2})$, it is easy to see that $zf(z)$ vanishes as $\sim 1/z^2$ as $|z| \to \infty$, therefore, Arfken Equation 11.123 applies. This gives

$$2^4\mathcal{S} = \sum_j \underset{z\to z_j}{Res}\left[\pi\sec(\pi z)\frac{(-1)^z}{z^3}\right] = \sum_j \underset{z\to z_j}{Res}\ f(z) \ , \tag{59}$$

where the definition of $f(z)$ is inferred from the equation, and z_j is the location of the pole. The residues of $f(z)$ can be found using Arfken Equation 11.68,

$$B_j = \frac{1}{(n-1)!}\lim_{z\to z_j}\left[\frac{d^{n-1}}{dz^{n-1}}\left((z-z_j)^n f(z)\right)\right] \ , \tag{60}$$

where n is the order of the pole. The function $f(z)$ has a pole of order three at $z=0$, so the above equation simplifies to

$$B_0 = \lim_{z\to 0}\frac{\pi}{2}\frac{d^2}{dz^2}\sec(\pi z) = \frac{\pi}{2}\left[\pi^2\sec^3(\pi z) + \pi^2\tan^2(\pi z)\sec(\pi z)\right]_{z=0} = \frac{\pi^3}{2}[1+0] \ , \tag{61}$$

however, there is also poles at $z = (k+\frac{1}{2})/2$, for all integer k, from the secant factor. These are all simple poles so their residues are given by

$$\begin{aligned} B_k &= \lim_{z\to(k+\frac{1}{2})/2}\frac{\pi[z-(k+\frac{1}{2})/2]}{z^3\cos(\pi z)} = \lim_{z\to(k+\frac{1}{2})/2}\frac{\pi}{3z^2\cos(\pi z) - \pi z^3\sin(\pi z)} \\ &= \frac{-1}{[(k+\frac{1}{2})/2]^3\sin[\pi(k+\frac{1}{2})/2]} = -\frac{1}{2^3}\frac{(-1)^k}{[k+\frac{1}{2}]^3}. \end{aligned}$$

Therefore the sum of all B_k's is

$$\sum_{k=-\infty}^{\infty} B_k = -\sum_{k=-\infty}^{\infty}\frac{1}{2^3}\frac{(-1)^k}{[k+\frac{1}{2}]^3} = -2\mathcal{S} \ , \tag{62}$$

which gives the result of the sum to be

$$2^4\mathcal{S} = \frac{\pi^3}{2} - 2\mathcal{S} \quad\Rightarrow\quad 2\mathcal{S}(2^3+1) = \frac{\pi^3}{2} \quad\Rightarrow\quad \mathcal{S} = \frac{\pi^3}{2^2}\frac{1}{(2^3+1)} = \frac{\pi^3}{36} \ . \tag{63}$$

6 Arfken 11.9.7.

Show that

$$\frac{1}{\cosh(\pi/2)} - \frac{1}{3\cosh(3\pi/2)} + \frac{1}{5\cosh(5\pi/2)} - \ldots = \frac{\pi}{8} \ . \tag{64}$$

This sum takes the form

$$\mathcal{S} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \frac{1}{\cosh[(2n+1)\pi/2)]} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2(n+\frac{1}{2})} \frac{1}{\cosh[(n+\frac{1}{2})\pi)]} \ , \tag{65}$$

in order to use the contour integral based formulas for summations, this sum needs to be of the form $\sum_{n=-\infty}^{\infty}(-1)^n f(n+1/2)$, so to change the rage of the summation, it can be written

$$2\mathcal{S} = \sum_{n=-\infty}^{\infty} (-1)^n \frac{1}{2(n+\frac{1}{2})} \frac{1}{\cosh[(n+\frac{1}{2})\pi)]} \ , \tag{66}$$

where the summand can be written as a function of z,

$$f(z) \equiv \frac{1}{2z\cosh(\pi z)} \ . \tag{67}$$

Using the relations in Arfken Table 11.2 the sum can be written

$$2\mathcal{S} = \sum_j \underset{z\to z_j}{Res} \left[\frac{\pi \sec \pi z}{2z\cosh(\pi z)} \right] = \sum_j \lim_{z\to z_j} \left[(z - z_j) \frac{\pi \sec \pi z}{2z\cosh(\pi z)} \right] \tag{68}$$

$$= \sum_j \lim_{z\to z_j} \left[\frac{d}{dz} \left\{ \pi(z-z_j) \right\} / \frac{d}{dz} \left\{ 2z\cos(\pi z)\cosh(\pi z) \right\} \right] \tag{69}$$

$$= \sum_j \left[\frac{\pi}{2\cos(\pi z)[\pi z\sinh(\pi z) + \cosh(\pi z)] - 2\pi z \sin(\pi z)\cosh(\pi z)} \right]_{z=z_j} \ , \tag{70}$$

by using L'Hôpital's rule. The function $f(z)\pi\sec(\pi z)$ has a pole at $z = 0$ (from the secant) and poles at $z = (2k+1)/2 = k + \frac{1}{2}$, for all integers k (from the hyperbolic cosine). The pole at zero has residue B_0, given by the summand of Equation 70, evaluated at $z_j = 0$. For this value, $\sinh(0) = \sin(0) = 0$, while $\cosh(0) = \cos(0) = 1$. The other poles are given by the sum in Equation 70, now over k. For this z value, $\sin((k+\frac{1}{2})\pi) = (-1)^k$, while $\cos((k+\frac{1}{2})\pi) = 0$. Now Equation 70 becomes

$$2\mathcal{S} = B_0 + \sum_{k=-\infty}^{\infty} \frac{\pi}{-2\pi(k+\frac{1}{2})\sin(\pi(k+\frac{1}{2}))\cosh(\pi(k+\frac{1}{2}))} \tag{71}$$

$$= \frac{\pi}{2} - \sum_{k=-\infty}^{\infty} \frac{1}{2(k+\frac{1}{2})} (-1)^k \frac{1}{\cosh[\pi(k+\frac{1}{2})]} \tag{72}$$

$$= \frac{\pi}{2} - 2\mathcal{S} \tag{73}$$

$$\mathcal{S} = \frac{\pi}{8} \ . \tag{74}$$