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1 Arfken 7.2.5.

Consider the differential equation

$$m\frac{dv}{dt} = -kv^n \ , \tag{1}$$

where $v(t)$ is the velocity, and n is a positive integer. This can be converted to a differential equation for $v(x)$, where x is the position, by the chain rule

$$\frac{dv}{dx}\frac{dx}{dt} = -\frac{k}{m}v^n \ , \tag{2}$$

where $dx/dt = v$. These equations both allow separation of variables to be used:

$$dv\ v(t)^{-n} = -\frac{k}{m}dt \quad \text{and} \quad dv\ v(x)^{1-n} = -\frac{k}{m}dt \ . \tag{3}$$

Let the coefficient k/m be defined as α. Due to the change of sign of the exponent for different values of n, four different cases will be examined: $n = 0, n = 1, n = 2, n > 2$. The initial conditions for this system are $v(t = 0) = v_0$ and $x(t = 0) = 0$.

1.1 First case: $n = 0$.

The time-differential equation in Equation 3 becomes

$$\int_{v_0}^{v(t)} dv = -\alpha \int_0^t dt \quad \Rightarrow \quad v(t) = v_0 - \alpha t \ . \tag{4}$$

The position-differential equation in Equation 3 becomes

$$\int_{v_0}^{v(x)} v dv = -\alpha \int_0^x dx \quad \Rightarrow \quad v(x)^2 - v_0^2 = -2\alpha x \quad \Rightarrow \quad v(x) = \sqrt{v_0^2 - 2\alpha x} \ . \tag{5}$$

1.2 Second case: $n = 1$.

The time-differential equation in Equation 3 becomes

$$\int_{v_0}^{v(t)} \frac{dv}{v} = -\alpha \int_0^t dt \quad \Rightarrow \quad \ln v(t) - \ln v_0 = -\alpha t \ , \tag{6}$$

exponentiating both sides of the final equation makes the right hand side into $v(t)/v_0$, so the solution for $v(t)$ with $n = 1$ is

$$v(t) = v_o e^{-\alpha t} \ . \tag{7}$$

The position-differential equation in Equation 3 becomes

$$\int_{v_0}^{v(x)} dv = -\alpha \int_0^x dx \quad \Rightarrow \quad v(x) = v_0 - \alpha x \ . \tag{8}$$

1.3 Third case: $n = 2$.

The time-differential equation in Equation 3 becomes

$$\int_{v_0}^{v(t)} v^{-2}\ dv = -\alpha \int_0^t dt \quad \Rightarrow \quad -\left(\frac{1}{v(t)} - \frac{1}{v_0}\right) = -\alpha t \quad \Rightarrow \quad v(t) = \frac{1}{\alpha t + v_0^{-1}}\ . \tag{9}$$

The position-differential equation in Equation 3 becomes

$$\int_{v_0}^{v(x)} \frac{dv}{v} = -\alpha \int_0^x dx \quad \Rightarrow \quad v(x) = v_o e^{-\alpha x}\ . \tag{10}$$

just as in the $n = 1$ case for $v(t)$. Note that the solutions for $v_n(x)$ are the solutions for$v_{n-1}(t)$.

1.4 Fourth case: $n > 2$.

This is the general case, for all $n > 2$ both differential equations behave the same (not as each other, but as the respective equation for another n). In this case the time-differential equation is

$$\int_{v_0}^{v(t)} dv\ v^{-n} = -\alpha \int_0^t dt \quad \Rightarrow \quad \frac{1}{-n+1}(v(t)^{1-n} - v_0^{1-n}) = -\alpha t\ , \tag{11}$$

solving for $v(t)$ yields the solution

$$v(t) = [(n-1)\alpha t + v_0^{1-n}]^{1/(1-n)}\ . \tag{12}$$

The position-differential equation is now

$$\int_{v_0}^{v(x)} dv\ v^{1-n} = -\alpha t \quad \Rightarrow \quad v(t)^{2-n} - v_0^{2-n} = (n-2)\alpha t\ , \tag{13}$$

giving the solution

$$v(t) = [(n-2)\alpha t + v_0^{2-n}]^{1/(2-n)} \tag{14}$$

2 Arfken 7.2.17.

Consider the ordinary differential equation

$$(xy^2 - y)\ dx + x\ dy = 0\ , \tag{15}$$

which is an isobaric equation as described in Arfken Example 7.2.4. Using this method, the substitution $v = xy$ will make the ODE above separable. Note that

$$dv = xdy + ydx \quad \Rightarrow \quad dx = \frac{1}{y}(dv - \frac{v}{y}dy)\ . \tag{16}$$

Substituting in $x = v/y$ and dx from above, the ODE is

$$0 = (vy - y)\frac{1}{y}(dv - \frac{v}{y}dy) + \frac{v}{y}dy \tag{17}$$

$$= vdv - v^2dy - dv + \frac{v}{y}dy + \frac{v}{y}dy \tag{18}$$

$$= (v-1)dv + \frac{(2v - v^2)}{y}dy \tag{19}$$

$$= \frac{(v-1)}{(2v - v^2)}dv + \frac{dy}{y}\ . \tag{20}$$

Now define a function $f(v)$ equal to the denominator of the dv term. Then, its derivative is

$$\frac{df}{dv} = 2 - 2v = -\frac{1}{2}(v-1)\ . \tag{21}$$

Substituting this into Equation 20 makes the ODE

$$\frac{df}{dv}\frac{dv}{f} + \frac{dy}{y} = 0\ , \tag{22}$$

which can be integrated directly,

$$\ln[f(v)] + ln[y] = C'\ , \tag{23}$$

where C' is a constant of integration. After exponentiating both sides the solution is then

$$C = yf(v) = yf(xy) = y(2xy - xy^2) = xy^2(2-y)\ . \tag{24}$$

3 Arfken 7.3.4.

Consider the ODE
$$y'' + 2y' + 2y = 0 \ , \tag{25}$$
whose solution is of the form $y = e^{mx}$, as given by Arfken page 342. Plugging this form into the ODE yields
$$e^{mx}[m^2 + 2m + 2] = 0 \ . \tag{26}$$
The polynomial has roots $-1 \pm i$, so t he solution has the form
$$y \to e^{(-1+i)x} + e^{(-1-i)x} = e^{-x}(e^{ix} + e^{-ix}) \ . \tag{27}$$
Using the Euler trig identities the functional form of the solution is
$$y \to e^{-x}(\cos x + i \sin x + \cos x - i \sin x) = 2e^{-x} \cos x \ . \tag{28}$$
So the most general form of this solution is
$$y(x) = Ce^{-x} \cos(x + \delta) \ , \tag{29}$$
where C is an arbitrary constant and δ is an arbitrary phase.

4 Arfken 7.4.2.

Laguerre's equation is given by $xy'' + (1-x)y' + \alpha y = 0$, which in the form required to study singular points is

$$y'' + \frac{1-x}{x}y' + \frac{\alpha}{x} = 0 \ . \tag{30}$$

Classification of singular points depend on the functions P and Q, which for this ODE are

$$P(x) = \frac{1-x}{x} \qquad Q(x) = \frac{\alpha}{x} \ . \tag{31}$$

Note that $x = 0$ is a regular singular point because it diverges but,

$$(x-0)P(x) = 1 \quad \text{and} \quad (x-0)Q(x) = \alpha \ , \tag{32}$$

are both finite. The point as $x \to \infty$ must be examined to determine the behavior. Consider $z = 1/x$, so that as $x \to \infty$, $z \to 0$, then

$$P(z^{-1}) = \frac{1-\frac{1}{z}}{\frac{1}{z}} = z - 1 \quad \text{and} \quad Q(z^{-1}) = \alpha z \ . \tag{33}$$

The possibility of a singularity depends on the behavior of

$$\frac{2z - P(z^{-1})}{z^2} = \frac{z+1}{z^2} \quad \text{and} \quad \frac{Q(z^{-1})}{z^4} = \alpha z^3 \ , \tag{34}$$

as stated by Arfken page 344. Both of these expressions diverge as $z \to 0$ so that $x = \infty$ is an irregular, or essential, singular point.

5 Arfken 7.5.7.

Consider the differential equation, found through the quantum mechanical analysis of the Stark effect,

$$\frac{d}{d\xi}\left(\xi\frac{du}{d\xi}\right)+\left(\frac{1}{2}E\xi+\alpha-\frac{m^2}{4\xi}-\frac{1}{4}F\xi^2\right)u=0\ , \tag{35}$$

defining the coefficient of the u term as $G(\xi)$ and performing the derivative on the left allows this equation to be written as

$$u'+\xi u''+G(\xi)u=0\ . \tag{36}$$

A solution of the form

$$u(\xi)=\sum_{j=0}^{\infty}a_j\xi^{s+j}\ , \tag{37}$$

is assumed, and the derivatives follow. Plugging these into the ODE yield

$$\sum_{j=0}^{\infty}a_j(s+j)\xi^{s+j-1}+\xi\sum_{j=0}^{\infty}a_j(s+j)(s+j-1)\xi^{s+j-2}+G(z)\sum_{j=0}^{\infty}a_j\xi^{s+j}=0\ , \tag{38}$$

so the first two sums can be combined:

$$\sum_{j=0}^{\infty}a_j(s+j)[1+(s+j-1)]\xi^{s+j-1}+\frac{E}{2}\sum_{j=0}^{\infty}a_j\xi^{s+j+1}$$
$$+\alpha\sum_{j=0}^{\infty}a_j\xi^{s+j}-\frac{m^2}{4}\sum_{j=0}^{\infty}a_j\xi^{s+j-1}-\frac{F}{4}\sum_{j=0}^{\infty}a_j\xi^{s+j+2}=0\ . \tag{39}$$

The coefficient of the lowest power of ξ (ξ^{s+j-1} for $j=0$) must be zero, which gives rise to the indicial equation,

$$\left[a_j(s+j)^2a_j-\frac{m^2}{4}=0\right]_{j=0}\ , \tag{40}$$

we have the condition that a_0 is the first nonzero coefficient of the power series so the indicial equation becomes

$$s^2-\frac{m^2}{4}=0\ , \tag{41}$$

which has roots $s=\pm(m/2)$. Using the larger root, the power series solution about $\xi=0$ is

$$u(\xi)=\sum_{j=0}^{\infty}a_j\xi^{j+\frac{m}{2}}\ . \tag{42}$$

In the context of this problem m must be positive, and the definition of a_0 ensures it is nonzero. Taking the sum of the coefficients of each power of ξ in Equation 39 and setting the sum equal to zero,

$$\xi^{m/2}:a_1(\frac{m}{2}+1)(\frac{m}{2}+1)+\alpha a_0+a_1\frac{-m^2}{4}\quad\Rightarrow\quad a_1=-\frac{\alpha}{m+1}a_0 \tag{43}$$

$$\xi^{3m/2}:a_2(\frac{m}{2}+2)(\frac{m}{2}+2)+\frac{E}{2}a_0+\alpha a_1-\frac{m^2}{4}a_2\quad\Rightarrow\quad a_1=\frac{\left(-2\alpha^2+Em+E\right)}{4(m+1)(m+2)}a_0\ . \tag{44}$$

6 Arfken 7.5.10.

Consider the ODE

$$y'' + \frac{1}{x^2}y' - \frac{2}{x^2}y = 0 \ , \tag{45}$$

which is amenable to a power series solution of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{s+j} \ , \tag{46}$$

with obvious derivatives. Plugging this solution and its derivatives into the ODE yields

$$0 = \sum_{j=0}^{\infty} a_j(j+s)(j+s-1)x^{s+j-2} + \sum_{j=0}^{\infty} a_j(j+s)x^{s+j-3} - 2\sum_{j=0}^{\infty} a_j x^{s+j-2} \tag{47}$$

$$= \sum_{j=0}^{\infty} a_j[(j+s)(j+s-1)-2]x^{s+j-2} + \sum_{k=-1}^{\infty} a_{k+1}(k+1+s)x^{s+j-2} \ , \tag{48}$$

by combining the first and last sum and changing the index of summation such that $j = k+1$. The first term of the sum on the right can be pulled out (yielding the indicial equation) and the remaining two sums can be combined by changing the dummy index from k back to j,

$$0 = a_0 s x^{s-3} + \sum_{j=0}^{\infty} \Big[a_j[(j+s)(j+s-1)-2] + a_{j+1}(j+1+s) \Big] x^{s+j-2} \ . \tag{49}$$

By definition $a_0 \neq 0$, so $s = 0$, from the indicial equation. Therefore, the recursion relation that determines the coefficients a_j is

$$0 = a_j[(j)(j-1)-2] + a_{j+1}(j+1) \ . \tag{50}$$

This allows the coefficients to be determined:

$$j=0: \ 0 = a_0(-2) + a_1 \quad \Rightarrow \quad a_1 = 2a_0 \tag{51}$$

$$j=1: \ 0 = a_1(-2) + 2a_2 \quad \Rightarrow \quad a_2 = a_1 \tag{52}$$

$$j=2: \ 0 = a_2(0) + 2a_3 \quad \Rightarrow \quad a_3 = a_0 \ , \tag{53}$$

and therefore all subsequent a_j's are zero, so the power series truncates after the x^2 term. The solution is then

$$y = a_0(1 + 2x + 2x^2) \ . \tag{54}$$

7 Arfken 7.5.11.

The modified Bessel function $I_0(x)$ satisfies the differential equation

$$x^2 \frac{d^2}{dx^2} I_0(x) + x \frac{d}{dx} I_0(x) - x^2 I_0(x) = 0 \ . \tag{55}$$

The leading term of the asymptotic series representing this function is $e^x/\sqrt{2\pi x}$, which allows the function to be written as

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}} f(x) \ . \tag{56}$$

The derivatives of this function are

$$I_0' = \frac{e^x}{\sqrt{2\pi x}} \frac{(2xf' + (2x-1)f)}{2x} \tag{57}$$

$$I_0'' = \frac{e^x}{\sqrt{2\pi x}} \frac{(4x\left(xf'' + (2x-1)f'\right) + (4(x-1)x+3)f)}{4x^2} \ . \tag{58}$$

These can be plugged in to the ODE, and the entire equation can be divided through by $e^x/\sqrt{2\pi x}$, which yields

$$0 = \frac{1}{4}[4x\left(xf'' + (2x-1)f'\right) + (4(x-1)x+3)f] + \frac{1}{2}[2xf' + (2x-1)f] - x^2 f \ , \tag{59}$$

combining like terms in f yields

$$0 = x^2 f'' + (x + 2x^2 - x)f' + \left(x^2 - x + \frac{3}{4} + x - \frac{1}{2} - x^2\right) f = x^2 f'' + (2x^2)f' + \left(\frac{1}{4}\right) f \ . \tag{60}$$

The power series $b_0 + b_1 x^{-1} + b_2 x^{-2} + \ldots$ can be substituted into this ODE for $f(x)$,

$$0 = x^2 \sum_{j=0}^{\infty} b_j(-j)(-j-1)x^{-j-2} + (2x^2)\sum_{j=0}^{\infty} b_j(-j)x^{-j-1} + \left(\frac{1}{4}\right)\sum_{j=0}^{\infty} b_j x^{-j} \tag{61}$$

$$= \sum_{j=0}^{\infty} b_j \left[(-j)(-j-1) + \frac{1}{4}\right] x^{-j} + \sum_{j=0}^{\infty} 2b_j(-j)x^{-j+1} \ . \tag{62}$$

Changing the summation index on the second sum such that $j = k+1$ yields

$$0 = \sum_{j=0}^{\infty} b_j \left[(-j)(-j-1) + \frac{1}{4}\right] x^{-j} + \sum_{k=-1}^{\infty} 2b_{k+1}(-k-1)x^{-k} \ , \tag{63}$$

pulling out the first term of the second sum, and combining both sums yields

$$0 = \sum_{j=0}^{\infty} b_j \left[j(j+1) + \frac{1}{4} - 2b_{j+1}(j+1)\right] x^{-j} \ , \tag{64}$$

because the $k=-1$ term vanishes. This leads to the recursion relation

$$b_{j+1} = b_j \frac{j(j+1) + \frac{1}{4}}{2(j+1)} \ . \tag{65}$$

Given that the first term in the series expansion is 1, the following two terms are

$$b_1 = (1)\frac{0 + \frac{1}{4}}{2(1)} = \frac{1}{8} \quad \text{and} \quad b_2 = \left(\frac{1}{8}\right)\frac{(2) + \frac{1}{4}}{2(2)} \tag{66}$$