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Contents

1 Arfken 7.6.14.

The ODE

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \ , \tag{1}$$

is satisfied by $y_1(x)$ and has a second, linearly-independent solution,

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^s P(t)dt\right]}{[y_1(s)]^2} ds \ . \tag{2}$$

It is useful to note the Leibniz formula for the derivative of an integral is

$$\frac{d}{d\xi}\int_{g(\xi)}^{h(\xi)} f(x,\xi)dx = \int_{g(\xi)}^{h(\xi)} \frac{\partial f(x,\xi)dx}{\partial \xi} + f[h(\xi),\xi]\frac{\partial h(\xi)}{\partial \xi} - f[g(\xi),\xi]\frac{\partial g(\xi)}{\partial \xi} \ . \tag{3}$$

In order to show y_2 is a solution, the derivatives of y_2 must be found using the Leibniz formula. Considering Equation 2, define $\alpha(x)$ as the entire integral and $\beta(s)$ as the numerator inside the integral. This allows the equation to be written as $y_2 = y_1\alpha$, so the first derivative is

$$y_2' = y_1'\alpha + \alpha' y_1 \ , \tag{4}$$

where

$$\alpha' = \frac{d}{dx}\int^x \frac{\exp\left[-\int^s P(t)dt\right]}{[y_1(s)]^2} ds = \frac{d}{dx}\int^x \frac{\beta(s)}{[y_1(s)]^2} ds \ , \tag{5}$$

setting the lower bound to zero and using the Leibniz formula (with $\xi \to x$ and $x \to s$) this is

$$\alpha' = \int_0^x \frac{\partial}{\partial x}\frac{\beta(s)}{[y_1(x)]^2} ds + \frac{\beta(s)}{[y_1(x)]^2}\frac{\partial}{\partial x}x - 0 = \frac{\beta}{y_1^2} \ , \tag{6}$$

so

$$y_2' = y_1'\alpha + \frac{\beta}{y_1} \ . \tag{7}$$

This can be differentiated again,

$$y_2'' = y_1''\alpha + \alpha' y_1' + \frac{\beta'}{y_1} - \frac{\beta}{y_1^2}y_1' \ , \tag{8}$$

but from Equation 6 the second and fourth terms cancel, yielding

$$y_2'' = y_1''\alpha + \frac{\beta'}{y_1} \ . \tag{9}$$

Now the expression for the first derivative of β must be found,

$$\beta = \exp\left[-\int^s P(t)dt\right] \quad \Rightarrow \quad \beta' = \left[-\int^s P(t)dt\right]'\beta \ , \tag{10}$$

using the Leibniz formula this is

$$\beta' = -\beta\frac{d}{dx}\left[\int^s P(t)dt\right] = -\beta\left\{\int^s \frac{\partial}{\partial x}P(t)dt + P(x)\frac{d}{dx}s - 0\right\} \ , \tag{11}$$

but because the first term is not dependent on x it is zero and the second term is just P(x) is the coordinate s is renamed x. So the derivatives of y_2 are

$$y_2 = y_1\alpha \tag{12}$$

$$y_2' = y_1'\alpha + \frac{\beta}{y_1} \tag{13}$$

$$y_2'' = y_1''\alpha + \frac{-\beta P(x)}{y_1} \; . \tag{14}$$

To verify that y_2 is a solution, these derivatives can be inserted into Equation 1:

$$0 = y_2'' + P(x)y_2' + Q(x)y_2 \tag{15}$$

$$= y_1''\alpha + \frac{-\beta P(x)}{y_1} + y_1'\alpha P(x) + \frac{\beta}{y_1}P(x) + Q(x)y_1\alpha \tag{16}$$

$$= y_1''\alpha + y_1'\alpha P(x) + Q(x)y_1\alpha \tag{17}$$

$$= \alpha[y_1'' + P(x)y_1' + Q(x)y_1] = 0 \; , \tag{18}$$

because $y_1(x)$ is a solution to the original ODE. Therefore $y_2(x)$ is a solution as well.

2 Arfken 7.6.22 and 7.6.23.

The Chebyshev equation is given by

$$(1-x^2)y'' - xy' + n^2 y = 0 \quad \Rightarrow \quad y'' + \frac{-x}{1-x^2}y' + \frac{n^2}{1-x^2}y = 0 \ , \tag{19}$$

where n is an integer.

2.1 Arfken 7.6.22: Solutions for $n = 0$.

One solution to the Chebyshev equation for $n = 0$ is $y_1(x) = 1$. Arfken Equation 7.67 gives the formula for finding a second, linearly-independent solution,

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^{x_2} P(x_1)dx_1\right]}{[y_1(x_2)]^2} dx_2 \ , \tag{20}$$

where $P(x)$ is determined by the differential equation in the form

$$y'' + P(x)y' + Q(x)y = 0 \tag{21}$$

in this case, $P(x) = -x/(1-x^2)$. Therefore the second solution is given by

$$y_2(x) = \int^x \exp\left[-\int^{x_2} \frac{-s}{1-s^2} ds\right] dx_2 \ . \tag{22}$$

The integral in the exponential can be evaluated by defining $f = 1 - s^2$, so that $df = -2sds$, so

$$\int^{x_2} \frac{-sds}{1-s^2} = \int^{x_2} \frac{\frac{1}{2}df}{f} = \frac{1}{2}\ln[1-s^2]|_{x_2} = \frac{1}{2}\ln\left[1-x_2^2\right] \ . \tag{23}$$

This makes the second solution

$$y_2(x) = \int^x \exp\left(-\frac{1}{2}\ln\left[1-x_2^2\right]\right) dx_2 = \int^x (1-x_2^2)^{-\frac{1}{2}} dx_2 = \arcsin x \ . \tag{24}$$

Compare this to the solution found by direct integration of Equation 20,

$$(1-x^2)y_i'' - xy_i' = 0 \ . \tag{25}$$

Notice the second order equation does not contain any term with y_i, so the new function z can be defined as the first derivative of y_i. This makes the equation into an equivalent first order equation,

$$(1-x^2)z' = xz \ , \tag{26}$$

which can be solved through direct integration,

$$\frac{z'}{z} = \frac{x}{1-x^2} \quad \Rightarrow \quad \int \frac{dz}{z} = \int \frac{x}{1-x^2} dx \ . \tag{27}$$

Using a similar transformation as in Equation 23, this becomes

$$\ln z = -\frac{1}{2}\ln[1-x^2] \quad \Rightarrow \quad z = [1-x^2]^{-1/2} \ , \tag{28}$$

after exponentiating. Now the first derivative of y_i is known (z), so the solution is

$$y_i = \int z dx = \int [1-x^2]^{-1/2} dx = \arcsin x \ , \tag{29}$$

which is exactly the second solution found previously.

2.2 Arfken 7.6.23: Solutions for $n = 1$.

A solution to Equation 20 for $n = 1$ is $y_1(x) = x$. The $P(x)$ function for this value of n is the same as above, so the same method for finding y_2 can be applied. Now Equation 24 becomes

$$y_2(x) = x \int^x \frac{(1-s^2)^{-1/2}}{s^2} ds = x \left[-\frac{\sqrt{1-x^2}}{x} \right] , \tag{30}$$

using MATHEMATICA. So the second solution, found using the Wronskian double integral, is

$$y_2(x) = -(1-x^2)^{1/2} . \tag{31}$$

3 Arfken 7.7.4.

Consider the inhomogeneous ODE

$$y'' - 3y' + 2y = \sin x \ , \tag{32}$$

which has a solution of the form

$$y = C_1 y_1 + C_2 y_2 + y_p \ , \tag{33}$$

where C_1 and C_2 are constants, y_1 and y_2 are the two solutions to the related homogeneous equation, and y_p is the particular solution for the inhomogeneous equation. Notice the homogeneous equation is amenable to a solution of the form $e^{\alpha x}$. Using that as the ansatz, the homogeneous equation reduces to

$$\alpha^2 e^{\alpha x} - 3\alpha e^{\alpha x} + 2e^{\alpha x} = 0 \ , \tag{34}$$

after dividing out the exponential in each term, this equation has roots at $\alpha = 1, 2$. Which makes the solutions to the homogeneous equation,

$$y_1(x) = e^x \quad \text{and} \quad y_2(x) = e^{2x} \ . \tag{35}$$

Using variation of parameters, the particular solution can be written as

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \ . \tag{36}$$

This and its first derivative reduce to a simultaneous system of algebraic equations given by Arfken Equation 7.98. For the above solutions, they are

$$0 = e^x u_1' + e^{2x} u_2' \tag{37}$$

$$\sin x = e^x u_1' + 2e^{2x} u_2' \ , \tag{38}$$

the first can be solved for u_1' and plugged into the second, yielding

$$e^x(-e^x u_2') + 2e^{2x} u_2' = \sin x \tag{39}$$

$$e^{2x} u_2' = \sin x \tag{40}$$

$$u_2' = e^{-2x} \sin x \quad \Rightarrow \quad u_1' = -e^{-x} \sin x \ . \tag{41}$$

These can both be integrated with respect to x to find the coefficients of the homogeneous solutions in the particular solution,

$$u_1 = \int -e^{-x} \sin x dx = \frac{1}{2} e^{-x} (\sin x + \cos x) \tag{42}$$

$$u_2 = \int e^{-2x} \sin x dx = -\frac{1}{5} e^{-2x} (2\sin x + \cos x) \ , \tag{43}$$

which makes the particular solution,

$$y_p = e^x \frac{1}{2} e^{-x} (\sin x + \cos x) + e^{2x} [-\frac{1}{5} e^{-2x} (2\sin x + \cos x)] = \frac{1}{10} (\sin x + 3\cos x) \ . \tag{44}$$

Collecting all the parts, the general solution to Equation 32 is

$$y(x) = C_1 e^x + C_2 e^{2x} + \frac{1}{10} (\sin x + 3\cos x) \ . \tag{45}$$

4 Arfken 10.1.3.

Consider the boundary condition problem

$$-\frac{d^2y}{dx^2} - \frac{y}{4} = f(x) \quad \begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases} . \tag{46}$$

Which using the differential operator $\mathcal{L}$ is

$$\mathcal{L}y = \left\{ \frac{d}{dx}\left(-\frac{d}{dx}\right) + \left(-\frac{1}{4}\right) \right\} y = f(x) \tag{47}$$

To find the solution, first examine the homogeneous equation $y'' = -\frac{1}{4}y$, which is a simple oscillator with solutions of the form

$$y_1(x) = \sin(x/2) \quad \text{and} \quad y_2(x) = \cos(x/2) , \tag{48}$$

note the coefficients will be added in later. The Green's function, as defined by Arfken Equation 10.18, for these boundary conditions is

$$G(x,t) = \begin{cases} x < t: \ G_1(x,t) = y_1(x)h(t) = \sin(x/2)h_1(t) \\ x > t: \ G_2(x,t) = y_2(x)h(t) = \cos(x/2)h_2(t) \end{cases} , \tag{49}$$

which satisfy $G(0,t) = G(\pi,t) = 0$. Now the imposing the continuity of the Green's function at $x = t$,

$$y_1(t)h_1(t) = y_2(t)h_2(t) \tag{50}$$

$$\sin(t/2)h_1(t) = h_2(t)\cos(t/2) , \tag{51}$$

so $h_1(t) = A\cos(t/2)$ and $h_2(t) = A\sin(t/2)$. From Arfken Equation 10.19, the first derivative of the Green's function must have a discontinuity at $x = t$ equal to $1/p(x) = -1$, so the constant from above is

$$A = \left\{ -1 \left[\sin(x/2)\left(-\frac{1}{2}\sin(x/2)\right)\sin(x/2) - \left(\frac{1}{2}\cos(x/2)\right)\cos(x/2) \right] \right\}^{-1} \tag{52}$$

$$= \left\{ \frac{1}{2}(\cos^2(t/2) + \sin^2(t/2) \right\}^{-1} = 2 . \tag{53}$$

Therefore the Green's function for this boundary condition problem is

$$G(x,t) = \begin{cases} 2\sin(x/2)\cos(t/2) & 0 \le x \le t \\ 2\cos(x/2)\sin(t/2) & t \le x \le \pi \end{cases} . \tag{54}$$

5 Problem #5.

Consider the ODE $x^3 y'' = y$, with solutions of the from $y = e^{S(x)}$. It has previously been shown that the asymptotic behavior of the function in the exponential as $x \to 0$ is

$$S(x) = \frac{2}{\sqrt{x}} + \frac{3}{4}\ln x + D(x) \ , \tag{55}$$

where $D(x) \ll \ln x$ as $x \to 0$. Using this asymptotic behavior, the derivatives must obey

$$D' \ll \frac{1}{x} \quad \Rightarrow \quad (D')^2 \ll \frac{1}{x^2} \tag{56}$$

$$D'' \ll \frac{1}{x^2} \ . \tag{57}$$

5.1 Asymptotic Behavior as $x \to 0$.

The form of $D(x)$ must be a constant plus a function of x: $D(x) = \delta + \delta(x)$, with $\delta \ll 1$ as $x \to 0$. Using the ansatz for y results in the ODE $x^3[(S')^2 + S''] = 1$, which plugging in the form of $S(x)$ above gives

$$1 = x^3\left[\left(-x^{-3/2} + \frac{3}{4}\frac{1}{x} + D'\right)^2 + \frac{3}{2}x^{-5/2} - \frac{3}{4}\frac{1}{x^2} + D''\right] \tag{58}$$

$$= x^3\left[-2D'x^{-3/2} + (D')^2 + \frac{3}{2}D'x^{-1} - \frac{3}{2}x^{-5/2} + x^{-3} + \frac{9}{16}x^{-2} + \frac{3}{2}x^{-5/2} - \frac{3}{4}x^{-2} + D''\right] \tag{59}$$

$$= x^3\left[-2D'x^{-3/2} + (D')^2 + \frac{3}{2}D'x^{-1} + x^{-3} + \frac{9}{16}x^{-2} - \frac{3}{4}x^{-2} + D''\right] \tag{60}$$

$$0 = x^3\left[-2D'x^{-3/2} + (D')^2 + \frac{3}{2}D'x^{-1} + \frac{9}{16}x^{-2} - \frac{3}{4}x^{-2} + D''\right] \ . \tag{61}$$

This can be simplified using the asymptotic relations listed above. The second term and the sixth term are negligible compared to the fourth term. Additionally, if $D' \ll x^{-1}$ then $D'x^{-1} \ll x^{-2}$, so that term can be neglected as well. Combining the remaining x^{-2} terms this is

$$0 \sim x^3\left[-2D'x^{-3/2} - \frac{3}{16}x^{-2}\right] \ , x \to 0 \ . \tag{62}$$

Pulling out a factor of $x^{-3/2}$ from the brackets and writing the asymptotic relation for the two terms in the bracket is

$$D' \sim -\frac{3}{32}x^{-1/2}, \ x \to 0, \tag{63}$$

and integrating once,

$$D \sim -\frac{3}{16}x^{1/2} + \delta, \ x \to 0, \tag{64}$$

where δ was picked up as a constant of integration. Noting that $\sqrt{x} \ll \ln x$ as $x \to 0$ (absolute value of log is much larger), and additionally, $x^{1/2} \ll 1$, this form of D obeys the restrictions on $D(x)$. Therefore the full solution is

$$S(x) = \frac{2}{\sqrt{x}} + \frac{3}{4}\ln x - \frac{3}{16}x^{1/2} + \delta \ . \tag{65}$$

5.2 Power Series Solution.

The solution to the ODE above can be written

$$y(x) = Kx^{3/4}e^{2/\sqrt{x}}w(x)\ , \tag{66}$$

where K is some constant. The first and second derivatives of y (found in Mathematica) are

$$y'(x) = \frac{e^{2/\sqrt{x}}}{4x^{3/4}}K\left[(-4+3\sqrt{x})w + 4x^{3/2}w'\right] \tag{67}$$

$$y''(x) = \frac{e^{2/\sqrt{x}}}{16x^{9/4}}K\left[(16-3x)w + 8(-4x^{3/2}+3x^2)w' + 16x^3w''\right]\ . \tag{68}$$

These can be plugged into the original ODE, $x^3y'' = y$, and noting that the exponential part and the constant K cancel on each side this is

$$x^{3/4}w = \frac{x^3}{16x^{9/4}}\left[(16-3x)w + 8(-4x^{3/2}+3x^2)w' + 16x^3w''\right] \tag{69}$$

$$16w = (16-3x)w + 8(-4x^{3/2}+3x^2)w' + 16x^3w'' \tag{70}$$

$$\frac{16w}{16x^3} = (16-3x)\frac{w}{16x^3} + (-4x^{3/2}+3x^2)\frac{8w'}{16x^3} + w'' \tag{71}$$

$$\frac{w}{x^3} = \frac{w}{x^3} - \frac{3w}{16x^2} + \left(-\frac{2}{x^{3/2}} + \frac{3}{2x}\right)w' + w''\ , \tag{72}$$

therefore $w(x)$ satisfies the equation

$$w'' + \left(\frac{3}{2x} - \frac{2}{x^{3/2}}\right)w' - \frac{3}{16x^2}w = 0\ . \tag{73}$$

Now assume a power series for $w(x)$,

$$w(x) \sim \sum_{n=0}^{\infty} a_n x^{n/2} \quad\Rightarrow\quad w'(x) \sim \sum_{n=0}^{\infty} a_n\frac{n}{2}x^{(n-2)/2} \quad\Rightarrow\quad w''(x) \sim \sum_{n=0}^{\infty} a_n\frac{n(n-2)}{4}x^{(n-4)/2}\ , \tag{74}$$

let $a_0 = 1$. Plugging these expressions into Equation 73 yields the expression

$$0 = \sum_{n=0}^{\infty} a_n\frac{n(n-2)}{4}x^{(n-4)/2} + \sum_{n=0}^{\infty}\left(\frac{3}{2x} - \frac{2}{x^{3/2}}\right)a_n\frac{n}{2}x^{(n-2)/2} - \sum_{n=0}^{\infty}\frac{3}{16x^2}a_nx^{n/2} \tag{75}$$

$$= \sum_{n=0}^{\infty} a_n\frac{n(n-2)}{4}x^{(n-4)/2} + \sum_{n=0}^{\infty} a_n\frac{3n}{4}x^{(n-4)/2} - \sum_{n=0}^{\infty} a_n n x^{(n-5)/2} - \sum_{n=0}^{\infty} a_n\frac{3}{16}x^{(n-4)/2} \tag{76}$$

$$= \sum_{n=0}^{\infty} a_n\left[\frac{n(n-2)}{4} + \frac{3n}{4} - \frac{3}{16}\right]x^{(n-4)/2} - \sum_{n=0}^{\infty} a_n n x^{(n-5)/2}\ . \tag{77}$$

The terms in the brackets in the first sum can be simplified to

$$\frac{1}{16}[4n(n-2) + 12n - 3] = \frac{1}{16}[4n^2 + 4n - 3]\ . \tag{78}$$

Now a change of variables $m = n - 5$ can be performed such that Equation 77, after multiplying through by an x^2 factor, becomes

$$0 = \frac{1}{16}\sum_{m=-5}^{\infty} a_{m+5}\left[4(m+5)^2 + 4(m+5) - 3\right]x^{m+1} - \sum_{m=-5}^{\infty} a_{m+5}(m+5)x^m \tag{79}$$

$$= \frac{1}{16}\sum_{m=-5}^{\infty} a_{m+5}\left[4m^2 + 44m + 117\right]x^{m+1} - \sum_{m=-5}^{\infty} a_{m+5}(m+5)x^m \ . \tag{80}$$

A change of variables can be done on the first sum to change the power of x again to be the same as the other sum. Let $p = m + 1$, which makes the above expression

$$0 = \frac{1}{16}\sum_{p=-4}^{\infty} a_{p+4}\left[4(p-1)^2 + 44(p-1) + 117\right]x^{p} - \sum_{m=-5}^{\infty} a_{m+5}(m+5)x^m \tag{81}$$

$$= \frac{1}{16}\sum_{p=-4}^{\infty} a_{p+4}\left[77 + 36p + 4p^2\right]x^{p} - \sum_{m=-5}^{\infty} a_{m+5}(m+5)x^m \tag{82}$$

$$= \frac{1}{16}\sum_{m=-4}^{\infty} a_{m+4}\left[77 + 36m + 4m^2\right]x^{m} - \sum_{m=-5}^{\infty} a_{m+5}(m+5)x^m \ , \tag{83}$$

by renaming the index p as m. Another change of variables can be done on both sums such that $n = m + 4$, yielding

$$0 = \frac{1}{16}\sum_{n=0}^{\infty} a_n\left[77 + 36(n-4) + 4(n-4)^2\right]x^{n-4} - \sum_{n=-1}^{\infty} a_{n+1}(n+1)x^{n-4} \tag{84}$$

$$= \frac{1}{16}\sum_{n=0}^{\infty} a_n\left[4n^2 + 4n - 3\right]x^{n-4} - \sum_{n=0}^{\infty} a_{n+1}(n+1)x^{n-4} \ , \tag{85}$$

in the last line, the $n = -1$ term was pulled out of the sum, moving the lower bound on the index of summation to 0. This term is zero because of the $n + 1$ factor. This makes both sums over the same range with the same exponent, so they can be combined to one sum,

$$0 = \sum_{n=0}^{\infty} x^{n-4}\left(\frac{1}{16}a_n\left[4n^2 + 4n - 3\right] - a_{n+1}[n+1]\right) \ . \tag{86}$$

Since every term in the sum will have different powers of x, the coefficient of each power of x must be zero in order to make the entire sum zero. This gives the recursion relation used to find the coefficients in the power series for $w(x)$,

$$a_{n+1} = \left(\frac{4n^2 + 4n - 3}{16(n+1)}\right)a_n \ . \tag{87}$$

In this form we can se the ratio of successive terms a_{n+1}/a_n is greater than one for large n. Thus, using the ratio test this is a divergent series, so the radius of convergence is zero.

6 Problem #6.

Consider the differential equation $y'' = y/x^5$ with the purpose of exploring the leading asymptotic behavior of the solutions as $x \to 0$. Assume a solution of the form $y = e^{S(x)}$, so the differential equation reduces to

$$S''e^S + (S')^2 e^2 = \frac{e^S}{x^5} \quad \Rightarrow \quad x^5[S'' + (S')^2] = 1 \ . \tag{88}$$

Now, make the assumption that near $x = 0$, $U'' \gg (U')^2$. Now the asymptotic behavior of Equation 88 is

$$x^5 S'' \sim 1, x \to 0 \quad \Rightarrow \quad \pm S' \sim x^{-5/2}, x \to 0 \ . \tag{89}$$

Therefore the asymptotic behavior of S as $x \to 0$ is $S \sim \pm\frac{2}{3}x^{-3/2}$. The asymptotic solution may now be expanded past leading order and written as

$$S(x) = \pm\frac{2}{3}x^{-3/2} + C(x) \quad \text{so} \quad S' = \mp x^{-5/2} + C', \quad S'' = \pm\frac{5}{2}x^{-7/2} + C'' \ . \tag{90}$$

Plugging this into Equation 88 yields

$$1 = x^5\left[\pm\frac{5}{2}x^{-7/2} + C'' + \left(\mp x^{-5/2} + C'\right)^2\right] \tag{91}$$

$$= x^5\left[\pm\frac{5}{2}x^{-7/2} + C'' + x^{-5} + (C')^2 \mp 2x^{-5/2}C'\right] \tag{92}$$

$$0 = x^5\left[\pm\frac{5}{2}x^{-7/2} + C'' + (C')^2 \mp 2x^{-5/2}C'\right] \ , \tag{93}$$

after canceling the $x^5 x^{-5} = 1$ term. Knowing that the leading asymptotic behavior must outweigh the following terms, at all derivatives, $S^{(n)} \gg C^{(n)}$. This lets the following asymptotic relations to be written:

$$\begin{cases} C \ll -\frac{2}{3}x^{-3/2} \\ C' \ll \mp x^{-5/2} \ \Rightarrow \ (C')^2 \ll x^{-5} \\ C'' \ll \pm\frac{5}{2}x^{-7/2} \end{cases} , x \to 0 \ . \tag{94}$$

Using these relations, an asymptotic relation can be written. Noting that the C'' term is negligible compared to the first term, and $(C')^2$ is negligible to the term that was canceled earlier, this relation is

$$0 \sim x^5\left[\pm\frac{5}{2}x^{-7/2} \mp 2x^{-5/2}C'\right] = x^5(x^{-5/2})\left[\pm\frac{5}{2}x^{-1} \mp 2C'\right], \quad x \to 0. \tag{95}$$

From this, clearly as $x \to 0$, $C' \sim \frac{5}{4}x^{-1}$, so that $C \sim \frac{5}{4}\ln x$. This gives the leading asymptotic behavior of the solutions as $x \to 0$ to be

$$y \sim \exp\left[\pm\frac{2}{3}x^{-3/2} + \frac{5}{4}\ln x\right] = e^{\pm\frac{2}{3}x^{-3/2}}x^{5/4}, \quad x \to 0 \ . \tag{96}$$