Dylan J. Temples: Solution Set One

Northwestern University, Electrodynamics I
Wednesday, January 13, 2016

Contents

1 Problem #1: General Forms of Gauss' and Stokes' Theorems.

The particular importance of the integral theorems of Gauss and Stokes arise from the transition from a volume to a surface integral, and from a surface integral to a line integral, respectively. These integral theorems are special forms of more general theorems. The more general forms can be derived from the theorems of Gauss and Stokes by an appropriate choice of the vector **A**, in particular,

$$\int_V dV \nabla \circ = \int_{S(V)} d\mathbf{S} \circ \ , \quad \text{and} \tag{1}$$

$$\int_S (d\mathbf{S} \circ \nabla) \times = \oint_{C(S)} d\mathbf{r} \circ \ , \tag{2}$$

where $\circ$ represents an ordinary, dot, or cross product. These operators using dot products give rise to the theorems of Gauss and Stokes and are already proven.

1.1 Gauss' Theorem - Ordinary Product.

To prove the equivalent of Gauss' theorem for the ordinary product choose a vector **A** to be an arbitrary constant vector times an arbitrary function of the position vector

$$\int_V dV \nabla_{\mathbf{r}} \cdot \mathbf{k}\psi(\mathbf{r}) = \int_{S(V)} d\mathbf{S} \cdot \mathbf{k}\psi(\mathbf{r}) \ . \tag{3}$$

Since $\mathbf{k}$ is a constant vector, the product can be written $\nabla_{\mathbf{r}} \cdot \mathbf{k}\psi(\mathbf{R}) = \mathbf{k} \cdot \nabla_{\mathbf{r}}\psi(\mathbf{r})$, and the constant vector can be pulled out of the integral

$$\int_V dV \nabla_{\mathbf{r}} \cdot \mathbf{k}\psi(\mathbf{r}) = \mathbf{k} \cdot \int_V dV \nabla_{\mathbf{r}}\psi(\mathbf{r}) \ . \tag{4}$$

On the left hand side of Equation 3, the order of the product can be swapped and the constant vector pulled out of the integral,

$$\int_{S(V)} d\mathbf{S} \cdot \mathbf{k}\psi(\mathbf{r}) = \mathbf{k} \cdot \int_{S(V)} d\mathbf{S}\psi(\mathbf{r}) \ , \tag{5}$$

which gives the relation

$$\mathbf{k} \cdot \int_V dV \nabla_{\mathbf{r}}\psi(\mathbf{r}) = \mathbf{k} \cdot \int_{S(V)} d\mathbf{S}\psi(\mathbf{r}) \ , \tag{6}$$

which implies the two integrals must be equal. This is of the form of Equation 1 with an ordinary product operating on an arbitrary scalar field $\psi(\mathbf{r})$.

1.2 Gauss' Theorem - Cross Product.

Now select the vector **A** such that it is the cross product of an arbitrary constant vector and an arbitrary vector that may have functional dependence on the spatial variables. The divergence theorem then says

$$\int_V dV \nabla_{\mathbf{r}} \cdot [\mathbf{k} \times \mathbf{B}(\mathbf{r})] = \int_{S(V)} d\mathbf{S} \cdot [\mathbf{k} \times \mathbf{B}(\mathbf{r})] \ , \tag{7}$$

The $\nabla_{\mathbf{r}} \cdot \mathbf{A}$ term on the left hand side can be written as

$$\int_V dV[\mathbf{B} \cdot (\nabla_{\mathbf{r}} \times \mathbf{k}) - \mathbf{k} \cdot (\nabla_{\mathbf{r}} \times \mathbf{B})] = \int_{S(V)} d\mathbf{S} \cdot [\mathbf{k} \times \mathbf{B}(\mathbf{r})] \ , \tag{8}$$

the first term of the left hand side drops out because the curl of a constant vector is zero, so it becomes

$$- \int_V dV[\mathbf{k} \cdot (\nabla_{\mathbf{r}} \times \mathbf{B}) = -\mathbf{k} \cdot \int_V dV(\nabla_{\mathbf{r}} \times \mathbf{B}) \ , \tag{9}$$

because the constant vector can be moved outside the integral. Also, the right hand side becomes

$$\int_{S(V)} d\mathbf{S} \cdot \mathbf{k} \times \mathbf{B} = \mathbf{k} \cdot \mathbf{B} \times d\mathbf{S} = \mathbf{k} \cdot \int_{S(V)} \mathbf{B} \times d\mathbf{S} = -\mathbf{k} \cdot \int_{S(V)} d\mathbf{S} \times \mathbf{B} \ , \tag{10}$$

so

$$-\mathbf{k} \cdot \int_V dV(\nabla_{\mathbf{r}} \times \mathbf{B}) = -\mathbf{k} \cdot \int_{S(V)} d\mathbf{S} \times \mathbf{B} \ , \tag{11}$$

which implies the integral operators must be equal. These operators are of the form of Equation 1 with the circle replaced with a cross product, which proves the general case that these integral operators work with any type of product.

1.3 Stokes' Theorem - Ordinary Product.

Let the vector $\mathbf{A}$ be the same as in section 1.1, and note that

$$\nabla_{\mathbf{r}} \times \mathbf{a}\phi(\mathbf{r}) = \nabla\phi \times \mathbf{a} + \phi(\nabla \times \mathbf{a}) \ . \tag{12}$$

Therefore we can write, from Stokes' theorem

$$\int_S d\mathbf{S} \cdot \nabla \times \mathbf{k}\psi(\mathbf{r}) = \int_S d\mathbf{S} \cdot [\nabla\psi \times \mathbf{k} + \psi(\nabla \times \mathbf{k})] = \oint_{C(S)} d\mathbf{r} \cdot \mathbf{k}\psi(\mathbf{r}) \ , \tag{13}$$

note the curl of the constant vector is zero so the second term of the middle expression is zero. Through associativity of dot products, the right hand side can be written

$$\oint_{C(S)} d\mathbf{r} \cdot \mathbf{k}\psi = \oint_{C(S)} (\mathbf{k} \cdot d\mathbf{r})\psi = \mathbf{k} \cdot \oint_{C(S)} d\mathbf{r}\psi \ . \tag{14}$$

The middle expression in Equation 13 is

$$\int_S d\mathbf{S} \cdot \nabla\psi \times \mathbf{k} = \int_S \mathbf{k} \cdot d\mathbf{S} \times \nabla\psi = \mathbf{k} \cdot \int_S d\mathbf{S} \times \nabla\psi \ , \tag{15}$$

note that this integral operator is the only form of the left hand side of Equation 2 that results in a vector by using an ordinary product. Using these expressions, it is found that

$$\int_S d\mathbf{S} \times \nabla\psi = \oint_{C(S)} d\mathbf{r}\psi \ , \tag{16}$$

which proves Stokes' theorem for the ordinary product, for an arbitrary scalar field $\psi(\mathbf{r})$.

1.4 Stokes' Theorem - Cross Product.

Let the vector **A** be the same as in section 1.2. Similarly to section 1.3, the right hand side of Stokes' theorem becomes

$$\oint_{C(S)} d\mathbf{r} \cdot [\mathbf{k} \times \mathbf{B}(\mathbf{r})] = \oint_{C(S)} \mathbf{k} \cdot (\mathbf{B} \times d\mathbf{r}) = -\mathbf{k} \cdot \oint_{C(S)} d\mathbf{r} \times \mathbf{B} \ , \tag{17}$$

which has an integral operator equivalent to the right hand side of Equation 2 with a cross product. The left hand side of Stokes' theorem is

$$\int_S d\mathbf{S} \cdot \nabla_{\mathbf{r}} \times [\mathbf{k} \times \mathbf{B}(\mathbf{r})] \ , \tag{18}$$

but the vector product can be simplified to

$$\nabla_{\mathbf{r}} \times [\mathbf{k} \times \mathbf{B}(\mathbf{r})] = \mathbf{k}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{k}) + (\mathbf{B} \cdot \nabla)\mathbf{k} - (\mathbf{k} \cdot \nabla)\mathbf{B} \tag{19}$$

$$= \mathbf{k}(\nabla \cdot \mathbf{B}) - (\mathbf{k} \cdot \nabla)\mathbf{B} \ , \tag{20}$$

because any differential operator interacting with the constant vector **k** is zero. Using Einstein notation to write the dot products this is

$$\mathbf{k}(\nabla_i B_i) - (k_j \nabla_j)\mathbf{B} \ , \tag{21}$$

where ∇_k is the kth element of the differential operator $\nabla_{\mathbf{r}}$, taking the dot product with the surface element is

$$d\mathbf{S} \cdot [\mathbf{k}(\nabla_i B_i) - (k_j \nabla_j)\mathbf{B}] = d\mathbf{S} \cdot \mathbf{k}(\nabla_i B_i) - (k_j \nabla_j)\mathbf{B} \cdot d\mathbf{S} \ , \tag{22}$$

in Einstein notation,

$$dS_j k_j (\nabla_i B_i) - (k_j \nabla_j) B_i dS_i = k_J [dS_j \nabla_i B_i - \nabla_j B_i dS_i] \ . \tag{23}$$

Now the Einstein sum over the dummy variable i can be rewritten as dot products

$$k_j [dS_j (\nabla \cdot \mathbf{B}) - \nabla_j (\mathbf{B} \cdot d\mathbf{S})] \ , \tag{24}$$

similarly the sum over j, so finally we get

$$d\mathbf{S} \cdot \nabla_{\mathbf{r}} \times [\mathbf{k} \times \mathbf{B}(\mathbf{r})] = -\mathbf{k} \cdot [-d\mathbf{S}(\nabla \cdot \mathbf{B}) + \nabla(\mathbf{B} \cdot d\mathbf{S})] \tag{25}$$

$$= -\mathbf{k} \cdot [\nabla(\mathbf{B} \cdot d\mathbf{S}) - d\mathbf{S}(\nabla \cdot \mathbf{B})] \tag{26}$$

$$= -\mathbf{k} \cdot [\nabla(\mathbf{B} \cdot d\mathbf{S}) - d\mathbf{S}(\mathbf{B} \cdot \nabla)] \ . \tag{27}$$

On a side note, the vector triple product can be written as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} \ , \tag{28}$$

using the identity for the vector triple product, this becomes

$$\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} \tag{29}$$

$$\mathbf{c}(\mathbf{a} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) = (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} \ , \tag{30}$$

so if we plug in the appropriate vectors to make the left hand side of the above equation look like the terms in the brackets in Equation 27, we find

$$\nabla(\mathbf{B}\cdot d\mathbf{S}) - d\mathbf{S}(\mathbf{B}\cdot\nabla) = (d\mathbf{S}\times\nabla)\times\mathbf{B} \tag{31}$$

$$-\mathbf{k}\cdot[\nabla(\mathbf{B}\cdot d\mathbf{S}) - d\mathbf{S}(\mathbf{B}\cdot\nabla)] = -\mathbf{k}\cdot(d\mathbf{S}\times\nabla)\times\mathbf{B} = d\mathbf{S}\cdot\nabla_{\mathbf{r}}\times[\mathbf{k}\times\mathbf{B}(\mathbf{r})]\ , \tag{32}$$

which we can integrate over all space and equate to the right hand expression in Equation 17, due to Stokes' theorem and find

$$\int_S -\mathbf{k}\cdot(d\mathbf{S}\times\nabla)\times\mathbf{B} = -\mathbf{k}\cdot\int_S(d\mathbf{S}\times\nabla)\times\mathbf{B} = -\mathbf{k}\cdot\oint_{C(S)} d\mathbf{r}\times\mathbf{B} \tag{33}$$

$$\int_S(d\mathbf{S}\times\nabla)\times\mathbf{B} = \oint_{C(S)} d\mathbf{r}\times\mathbf{B}\ , \tag{34}$$

which is the exact form of Equation 2 with a cross product. This proves Equation 2 for all three types of products.

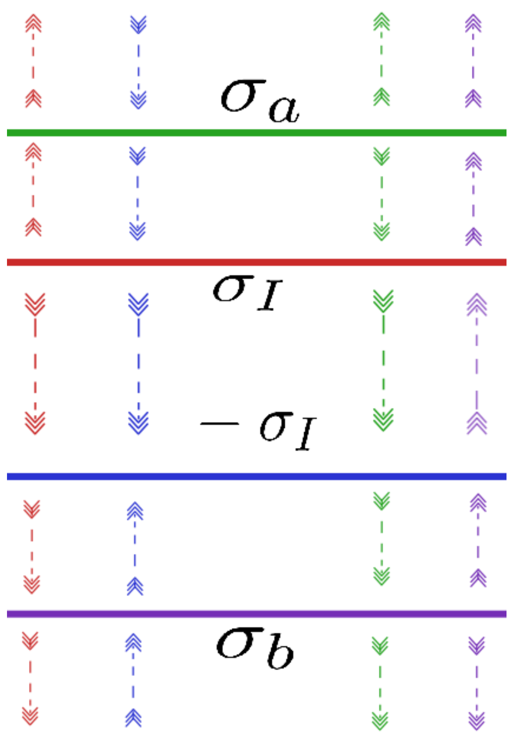

Figure 1: Diagram of electric field lines from four charged infinite planes, assuming the electric fields are not affected by the other charged planes. The direction of the green and purple arrows would reverse if the charge density was of opposite sign, but would still oppose each other.

2 Problem #2: Surface Charge Densities of Parallel Plate Capacitor.

Two infinite, conducting, plane sheets of uniform thicknesses t_1 and t_2, respectively, are placed parallel to one another with their adjacent faces separated by a distance L. The first sheet has a total charge per unit area (sum of surface charge densities on either side) equal to q_1, while the second has q_2.

This gives the relationships

$$q_1 = \sigma_{1I} + \sigma_{1E} \tag{35}$$

$$q_2 = \sigma_{2I} + \sigma_{2E} \ , \tag{36}$$

where σ_{jI} is the surface charge density of the interior face of the jth conductor, and σ_{jE} is the surface charge density of the external face.

2.1 Adjacent Faces.

It can be shown that the surface charge densities on the adjacent faces are equal and opposite. Consider a cylindrical Gaussian surface of radius a, where the ends are parallel to the plates and one end lies inside each conductor. Because these faces are inside the conductor, the electric field passing through these faces must be zero. Additionally, the electric field between the plates is purely in the axial direction, so no electric field lines pass through the surface of the Gaussian cylinder. Therefore, by Gauss' law,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enc}}{\epsilon_0} \quad \Rightarrow \quad 0 = \frac{1}{\epsilon_0}(\pi a^2 \sigma_{1I} + \pi a^2 \sigma_{2I}) = \sigma_{1I} + \sigma_{2I} \ . \tag{37}$$

So it must be that the surface charge densities of the interior faces are equal and opposite,

$$\sigma_{1I} = -\sigma_{2I} \ . \tag{38}$$

2.2 Exterior Faces.

It can be shown that the surface charge densities on the exterior faces of the two sheets are the same. To see this, it is important to enforce the electric field inside the conductor is zero. First consider four infinite planes that correspond to the faces of each conductor (the ones that are parallel to the electric field), the outer planes have charge density σ_a and σ_b and the inner planes have $\pm\sigma_I$. Assume the electric field from each plane can exist in each region around the planes, now in the region between the two surfaces of each plate (where the conductor bulk would be) the electric fields from the two inner plates exactly cancel because their charge densities are equal and opposite so the field lines in these regions would oppose and exactly cancel. Therefore in these regions only the electric field lines from the outer faces contribute. If we enforce the electric field in these regions must be zero then the contributions from the outer planes must cancel exactly, so they must have the same magnitude of charge density. It can be shown by this method, using the symmetry of the problem that the charge density must be the same sign as well to ensure the field lines cancel, see Figure 1; therefore

$$\sigma_{1E} = \sigma_{2E} \ . \tag{39}$$

In Figure 1 the green and purple field lines are due to the surface charge density of the exterior faces, clearly they must have the same sign for the electric fields to cancel inside the conductors.

The diagram shows $\sigma_a = \sigma_b > 0$, but would have the same effect for $\sigma_a = \sigma_b < 0$. Shortly, because the charge densities on the inner faces are equal and opposite, the charge density induced on one outer face is due only to the other outer face. Therefore, the induced charges must be exactly the same.

2.3 Plate Geometry.

Due to both plates being infinite in size, all horizontal components of the electric field exactly cancel, so the only net electric fields in this system are perpendicular to the plane. Consider a Gaussian cylinder as in section 2.1, but of any height. The only surfaces that contribute in Gauss' law are the ends, because the normal vector to the surface forming the barrel of the cylinder is always perpendicular to the electric field, so the dot product will always be zero,

$$E_t(\pi a^2) + E_b(\pi a^2) = \frac{Q_{enc}}{\epsilon_0} = \frac{\pi a^2 \sum \sigma_i}{\epsilon_0} \ , \tag{40}$$

where σ_i are the surface charge densities enclosed by the Gaussian surface and E_t and E_b are the magnitudes of the electric field passing through the top and bottom faces of the Gaussian cylinder, respectively. Clearly, the area of the face does not matter, so the electric field in any region does not depend on any geometric parameter of the system, but only on the surface charge density. Using the results of sections 2.1 and 2.2, these surface densities are clearly also independent of t_1, t_2, and L.

2.4 Electric Fields.

From the previous solutions and Equations 35 and 36, we have

$$\begin{cases} q_1 &= \sigma_I + \sigma_E \\ q_2 &= -\sigma_I + \sigma_E \end{cases} \quad \Rightarrow \quad \begin{cases} \sigma_E &= \frac{1}{2}(q_1 + q_2) \\ \sigma_I &= \frac{1}{2}(q_1 - q_2) \end{cases} \ . \tag{41}$$

Consider the five regions created by the four parallel infinite sheets of charge: A) vacuum outside first conductor, B) first conductor, C) vacuum between conductors, D) the second conductor, and E) vacuum outside second conductor. We can construct four Gaussian surfaces, each a pillbox of face area A (height doesn't matter due to there being no transverse electric field) with one face in the conductor and one face in the vacuum. The electric field is zero inside the conductor so only the face in the vacuum contributes to the surface integral. Therefore, from Gauss' law,

$$EA = \frac{\sigma_{enc} A}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\sigma_{enc}}{\epsilon_0} \ , \tag{42}$$

where σ_{enc} is the surface charge density enclosed in the Gaussian pillbox. The electric field in regions B and D must be zero, and the fields in the regions outside the conductors are due completely to the exterior faces surface charge density

$$E_A = E_E = \frac{\sigma_E}{\epsilon_0} \ , \tag{43}$$

and the electric field in region C has two contributions (that must oppose if charge densities are the same sign) so

$$E_C = \frac{\sigma_{I1}}{\epsilon_0} - \frac{\sigma_{I2}}{\epsilon_0} = \frac{\sigma_{I1}}{\epsilon_0} - \frac{-\sigma_{I1}}{\epsilon_0} = \frac{2\sigma_I}{\epsilon_0} \ . \tag{44}$$

Using Equation 41, we find that the electric fields are

$$E_{outside} = \frac{q_1 + q_2}{2\epsilon_0} \tag{45}$$

$$E_{inside} = \frac{q_1 - q_2}{\epsilon_0} \ , \tag{46}$$

which as stated previously do not depend on any distance scale (t_1, t_2, L).

2.4.1 A Special Case.

Consider the special case that $q_1 = -q_2 = Q$. From Equations 41, 45 and 46, it is easy to see that

$$\begin{cases} \sigma_E & = \frac{1}{2}[Q + (-Q)] = 0 \\ \sigma_I & = \frac{1}{2}[Q - (-Q)] = Q \end{cases} \quad \Rightarrow \quad \begin{cases} E_{outside} & = 0 \\ E_{inside} & = \frac{2Q}{\epsilon_0} \ , \end{cases} \ . \tag{47}$$

So the electric field outside the capacitor is zero and the field inside is proportional to twice the charge density.

3 Problem #3: Quantum Capacitance.

You have probably seen more than a few times the derivation of the capacitance C of a parallel plate capacitor, with a charge Q on one plate, and a corresponding potential difference V between its plates, with $C = Q/V$. What is glossed over in this simple analysis is how the potential is actually measured. Considering the case when the charges in question are electrons, measuring the voltage on a plate means equilibrating the electrochemical potential μ of the electrons on the plate with the electrochemical potential of the metal forming the electrical leads of a voltmeter. You know from your statistical physics courses that the electrochemical potential of fermions is determined by their number density.

Consider then a parallel plate capacitor of area S and separation d where one of the plates is made from a typical metal such as copper or gold, but the other plate is made from graphene, a single, two-dimensional atomic layer of carbon atoms with low charge carrier density. Applying a potential across this capacitor will draw charges to the capacitor plates. For a typical metal like copper, this would not represent a significant change in the charge density, but it does for a low density material like graphene, and hence will result in an appreciable change in electrochemical potential. This results in a correction C_Q to the geometric capacitance $C_g = \epsilon S/d$. By calculating this correction one can show how to obtain information about the density of states dn/dE of charge carriers in graphene by measuring the capacitance of this device. Here n is the areal charge density and E is the energy of the charge carriers.

Previously, it was shown that the charge density on the interior faces of a parallel plate capacitor must be equal an opposite, since all charge carriers have the same charge magnitude e, then the number density on both plates n must be equal. Since the charge carrier number density of copper is much higher than graphene, the graphene must draw additional charge carriers. If the graphene accepts N charge carriers, its charge raises by $Q = Ne$. Therefore the change in electrochemical potential (equivalent to the change in energy) is

$$\Delta\mu = \Delta E = \frac{N}{S\frac{dn}{dE}} = \frac{Q}{eS\frac{dn}{dE}} \ . \tag{48}$$

The change in voltage is given by the change in energy per unit charge

$$\Delta V = \frac{\Delta\mu}{e} = \frac{Q}{e^2 S\frac{dn}{dE}} \ . \tag{49}$$

By the definition of capacitance, we can find the change in capacitance of the whole capacitor, which in this case is exactly the capacitance due to the quantum mechanical behavior,

$$\Delta C = \frac{Q}{\Delta V} = e^2 S\frac{dn}{dE} \ , \tag{50}$$

so the total capacitance of the system is

$$C = C_g + C_q = \frac{\epsilon_0 S}{d} + e^2 S\frac{dn}{dE} \ . \tag{51}$$

4 Problem #4: Force due to Self-Capacitance.

Consider two capacitors with self capacitances C_1 and C_2. (For clarity, you may want to think of them as spherical conducting shells, as discussed in class, but this is not necessary.) They are placed such that the distance r between them is large in comparison to their sizes. Let V_1 bet the potential of the first conductor, and V_2 be the potential of the second conductor.

4.1 Charge on Conductors.

A single capacitor with capacitance C with charge Q has potential $V = Q/C$. Now a point charge Q' is placed at a distance r from the capacitor, which creates a potential $V = kQ'/r$, with $k^{-1} = 4\pi\epsilon_0$, at the location of the capacitor. Generalizing this to be two point-capacitors (in the sense the size of the physical capacitor is negligible compared to the distance between them r) the total potential on each capacitor is given by

$$V_1 = \frac{Q_1}{C_1} + \frac{k}{r}Q_2 \tag{52}$$

$$V_2 = \frac{Q_2}{C_2} + \frac{k}{r}Q_1 \ , \tag{53}$$

after multiplying by the respective capacitances,

$$C_1V_1 = Q_1 + \frac{k}{r}C_1Q_2 \tag{54}$$

$$C_2V_2 = Q_2 + \frac{k}{r}C_2Q_1 \ , \tag{55}$$

which can be rearranged into expressions for the charges,

$$Q_1 = C_1V_1 - \frac{k}{r}C_1Q_2 \tag{56}$$

$$Q_2 = C_2V_2 - \frac{k}{r}C_2Q_1 \ . \tag{57}$$

To find an expression for Q_2 plug Equation 56 into Equation 55, which yields

$$C_2V_2 = Q_2 + \frac{k}{r}C_2\left[C_1V_1 - \frac{k}{r}C_1Q_2\right] \ , \tag{58}$$

combining the terms with Q_2 and rearranging yields

$$Q_2\left[1 - \frac{k^2}{r^2}C_1C_2\right] = C_2V_2 - \frac{k}{r}C_1C_2V_1 \ , \tag{59}$$

and multiplying each side by k^2/r^2 gives

$$Q_2\left[\frac{r^2}{k^2} - C_1C_2\right] = \frac{C_2}{k}\left[\frac{V_2r}{k} - C_1V_1\right]r \ . \tag{60}$$

Which is an expression for the total charge on the second capacitor. An analogous equation can be found for the charge on the first by plugging Equation 57 into Equation 54 (or by arguing this must be the case by symmetry) yields the expression

$$Q_1\left[\frac{r^2}{k^2} - C_1C_2\right] = \frac{C_1}{k}\left[\frac{V_1r}{k} - C_2V_2\right]r \ . \tag{61}$$

With expressions for the charges on each capacitor known, the force between them can be calculated.

4.2 Repulsive Force.

The magnitude of the force between the two capacitors is simply given by the Coulomb force law

$$F = \frac{k}{r^2} Q_1 Q_2 = \frac{k}{r^2}\left(\frac{C_1 r}{k}\right)\left[\frac{V_1 r}{k} - C_2 V_2\right]\left(\frac{C_2 r}{k}\right)\left[\frac{V_2 r}{k} - C_1 V_1\right] \Bigg/ \left[\frac{r^2}{k^2} - C_1 C_2\right]^2 \tag{62}$$

$$= \frac{C_1 C_2}{k}\frac{(k^{-1} r V_1 - C_2 V_2)(k^{-1} r V_2 - C_1 V_1)}{[(k^{-1} r)^2 - C_1 C_2]^2} \tag{63}$$

$$= 4\pi\epsilon_0 C_1 C_2 \left[\frac{(4\pi\epsilon_0 r V_1 - V_2 C_2)(4\pi\epsilon_0 r V_2 - V_1 C_1)}{((4\pi\epsilon_0 r)^2 - C_1 C_2)^2}\right] . \tag{64}$$

5 Problem #5: Capacitance Matrix.

In class, we related the charge on the ith conductor in an ensemble of n conductors to the potentials ϕ_i on the other conductors by the equation

$$q_i = \sum_{j=1}^{n} C_{ij}\phi_j \ , \tag{65}$$

where C_{ij} are the so-called capacities of the system. Consider two concentric spherical shells of radii r_1 and r_2 ($r_2 > r_1$).

5.1 Coefficients of Capacity.

Let us define three regions: **Region I** interior to the small conductor ($0 \leq r_I < r_1$), **Region II** between the conductors ($r_1 < r_{II} < r_2$), and **Region III** exterior to the large conductor ($r_{III} > r_2$). The small and large spheres have charge q_1 and q_2, respectively. The electric field in each region can be easily found using Gauss' law. In **Region I** there is no charge internal to the Gaussian surface, so the electric field is zero. In **Region II** the electric field is only due to q_1, and in **Region III** the electric field is due to $q_1 + q_2$,

$$E_I = 0 \tag{66}$$

$$E_{II} = \frac{kq_1}{r^2} \tag{67}$$

$$E_{III} = \frac{k(q_1+q_2)}{r^2} \ . \tag{68}$$

Additionally the change in electric potential can be calculated in **Regions II** and **III**. Let ϕ_1 and ϕ_2 be the potentials of the small and large spheres, respectively. Then the potential drop in **Region II** is

$$\phi_2 - \phi_1 = -\int_{r_1}^{r_2} \mathbf{E}\cdot d\boldsymbol{\ell} = -\int_{r_1}^{r_2} \frac{kq_1}{r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}dr = -kq_1\left[-\frac{1}{r}\right]_{r_1}^{r_2} \tag{69}$$

$$= kq_1\left[\frac{1}{r_2} - \frac{1}{r_1}\right] = -kq_1\frac{r_2 - r_1}{r_1 r_2} \ . \tag{70}$$

The matrix elements C_{ij} are completely determined by the geometry of the system, independent of potentials and charges. Therefore, the potentials (or equivalently charges) can be fixed on each surface to values that make it simple to calculate the matrix elements, and the matrix elements will hold for any potential. Consider grounding the exterior shell and maintaining a voltage of ϕ_1 on the interior shell. Using Equation 67 yields

$$\phi_1 = kq_1\frac{r_2 - r_1}{r_1 r_2} \quad \Rightarrow \quad q_1 = \frac{r_1 r_2}{k(r_2 - r_1)}\phi_1 \ . \tag{71}$$

Using the definition of the capacities of the system

$$C_{ij} = \frac{\partial q_i}{\partial \phi_j} \ , \tag{72}$$

gives the result for the first matrix element,

$$C_{11} = \frac{r_1 r_2}{k(r_2 - r_1)} \ . \tag{73}$$

Similarly, we can set $0 = \phi_1 \neq \phi_2$, so that

$$\phi_2 = -kq_1 \frac{r_2 - r_1}{r_1 r_2} \quad \Rightarrow \quad q_1 = -\frac{r_1 r_2}{k(r_2 - r_1)} \phi_2 = -C_{11}\phi_1 \ , \tag{74}$$

so that $C_{12} = -C_{11}$.

Additionally, we can solve for the potential difference between the outer shell and the potential at infinity ϕ_∞, which will be set to zero. In this region,

$$\phi_\infty - \phi_2 = -\int_{r_2}^{\infty} \frac{k(q_1+q_2)}{r^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dr = -k(q_1+q_2) \int_{r_2}^{\infty} \frac{dr}{r^2} = k(q_1+q_2) \left[0 - \frac{1}{r_2} \right] \ , \tag{75}$$

which simplifies to

$$\phi_2 = \frac{k}{r_2}(q_1 + q_2) \ . \tag{76}$$

We can expand the matrix equation given by Equation 65, and using the relationship implied by Equation 74, to get the system of equations

$$q_1 = C_{11}(\phi_1 - \phi_2) \tag{77}$$

$$q_2 = C_{21}\phi_1 + C_{22}\phi_2 \ . \tag{78}$$

Now we can examine a third scenario in which the potential on the inner sphere is zero, and ϕ_2 on the outer sphere. From Equations 77 and 76, we have that

$$\phi_2 = \frac{k}{r_2}(-C_{11}\phi_2 + q_2) = \frac{kq_2}{r_2} - \frac{k}{r_2}\left(\frac{r_1 r_2}{k(r_2 - r_1)} \right) \phi_2 = \frac{kq_2}{r_2} - \left(\frac{r_1}{r_2 - r_1} \right) \phi_2 \tag{79}$$

$$\frac{kq_2}{r_2} = \phi_2 \left[1 + \frac{r_1}{r_2 - r_1} \right] = \phi_2 \left[\frac{r_2}{r_2 - r_1} \right] \ , \tag{80}$$

which gives the expression for the potential on the second sphere in this scenario

$$\phi_2 = q_2 \frac{k(r_2 - r_1)}{r_2^2} \ , \tag{81}$$

and using Equation 72, the coefficient of capacity is

$$C_{22} = \frac{q_2}{\phi_2} = \frac{r_2^2}{k(r_2 - r_1)} \ . \tag{82}$$

The final coefficient of capacity must be equal to the other off diagonal entry, due to pairwise interaction. The effect the inner sphere has on the outer must be the same as the outer sphere's effect on the inner. Therefore $C_{21} = C_{12} = -C_{11}$, so the whole capactiy matrix is

$$\hat{\mathbf{C}} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \frac{4\pi\epsilon_0 r_2}{r_2 - r_1} \begin{bmatrix} r_1 & -r_1 \\ -r_1 & r_2 \end{bmatrix} \tag{83}$$

5.2 Capacitance.

The inverse of Equation 65 is

$$\phi_j = \sum_{k=1}^{n} p_{jk} q_k \ , \tag{84}$$

where p_{jk} are the coefficients of potential, so that $\hat{\mathbf{p}} = \hat{\mathbf{C}}^{-1}$. The inverse of the matrix of the coefficients of capacity is

$$\hat{\mathbf{p}} = \frac{1}{4\pi\epsilon_0}\begin{bmatrix} \frac{1}{r_1} & \frac{1}{r_2} \\ \frac{1}{r_2} & \frac{1}{r_2} \end{bmatrix} \ . \tag{85}$$

Equation 84 can be expanded to two equations (one for ϕ_1 and one for ϕ_2) and subtracted to find an expression for the potential difference across the capacitor. Consider the potentials that arise from placing a charge $-Q$ on the inner conductor and a charge Q on the outer conductor. In this case, the system of equations is

$$\begin{cases} \phi_1 = p_{11}(-Q) + p_{12}(Q) \\ \phi_2 = p_{21}(-Q) + p_{22}(Q) \end{cases} \quad \rightarrow \quad \Delta V = V_2 - V_1 = (p_{11} + p_{22} - 2p_{12})Q \ , \tag{86}$$

and so from the definition of capacitance,

$$C = \frac{Q}{\Delta V} = \frac{1}{p_{11} + p_{22} - 2p_{12}} = \frac{1}{k}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r_2}\right]^{-1} = \frac{1}{k}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]^{-1} \tag{87}$$

$$= \frac{1}{k}\left[\frac{r_2 - r_1}{r_1 r_2}\right]^{-1} = 4\pi\epsilon_0 \frac{r_1 r_2}{r_2 - r_1} = C_{11} \ . \tag{88}$$