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Contents

1 Problem #1: Laplacian in Prolate Spheroidal Coordinates.

Consider the Dirichlet problem in prolate spheroidal (*i.e.*, ellipsoidal) coordinates (α, β, ϕ), related to the Cartesian coordinates (x, y, z) by the equations

$$x = c \sinh\alpha \sin\beta \cos\phi \quad y = c \sinh\alpha \sin\beta \sin\phi \quad z = c \cosh\alpha \cos\beta \ , \tag{1}$$

where

$$0 \leq \alpha < \infty, \quad 0 \leq \beta \leq \pi, \quad -\pi < \phi \leq \pi, \tag{2}$$

and $c > 0$. Using separation of variables, and assuming azimuthal (ϕ) symmetry, show that the solution of Laplace's equation in the interior of an ellipsoid can be written in the form

$$u(\alpha, \beta) = \sum_n A_n P_n(\cosh\alpha) P_n(\cos\beta) \ . \tag{3}$$

1.1 Derivation of Laplacian Operator.

The Laplacian in generalized curvilinear coordinates is defined by

$$\nabla^2 = \frac{1}{\prod_j h_j} \frac{\partial}{\partial q_i} \left[\frac{\prod_j h_j}{h_i^2} \frac{\partial}{\partial q_i} \right] = \frac{1}{h_\alpha h_\beta h_\phi} \frac{\partial}{\partial q_i} \left[\frac{h_\alpha h_\beta h_\phi}{h_i^2} \frac{\partial}{\partial q_i} \right] \tag{4}$$

where the sum over repeated indeces is implied, and h_i are the scale factors defined as

$$h_i = \sqrt{\sum_{k=1}^{3} \left(\frac{\partial x_k}{\partial q_i} \right)^2} \tag{5}$$

with $q_i \in \{\alpha, \beta, \phi\}$ and $x_k \in \{x, y, z\}$. Therefore, we find the scale factors are given by

$$h_\alpha = c\sqrt{(\cosh\alpha \sin\beta \cos\phi)^2 + (\cosh\alpha \sin\beta \sin\phi)^2 + (\sinh\alpha \cos\beta)^2} \tag{6}$$

$$h_\beta = c\sqrt{(\sinh\alpha \cos\beta \cos\phi)^2 + (\sinh\alpha \cos\beta \sin\phi)^2 + (-\cosh\alpha \sin\beta)^2} \tag{7}$$

$$h_\phi = c\sqrt{(-\sinh\alpha \sin\beta \sin\phi)^2 + (\sinh\alpha \sin\beta \cos\phi)^2} \ , \tag{8}$$

which we can examine case-by-case. Begin with the sacle factor for the first coordinate α,

$$\begin{aligned} h_\alpha &= c\sqrt{\cosh^2\alpha \sin^2\beta \cos^2\phi + \cosh^2\alpha \sin^2\beta \sin^2\phi + \sinh^2\alpha \cos^2\beta} \\ &= c\sqrt{\cosh^2\alpha \sin^2\beta + \sinh^2\alpha \cos^2\beta} \\ &= c\sqrt{\cosh^2\alpha \sin^2\beta + \sinh^2\alpha - \sinh^2\alpha \sin^2\beta} \\ &= c\sqrt{\sinh^2\alpha + (\cosh^2\alpha - \sinh^2\alpha)\sin^2\beta} \\ &= c\sqrt{\sinh^2\alpha + \sin^2\beta} \ , \end{aligned}$$

by using the identity $\cosh^2 x = 1 + \sinh^2 x$. The scale factor for ϕ is

$$\begin{aligned} h_\phi &= c\sqrt{\sinh^2\alpha \sin^2\beta \sin^2\phi + \sinh^2\alpha \sin^2\beta \cos^2\phi} \\ &= c \sinh\alpha \sin\beta \ , \end{aligned}$$

We can use a similar analysis as for h_α show the scale factor for β is

$$\begin{aligned}
h_\beta &= c\sqrt{\sinh^2\alpha\cos^2\beta\cos^2\phi + \sinh^2\alpha\cos^2\beta\sin^2\phi + \cosh^2\alpha\sin^2\beta} \\
&= c\sqrt{\sinh^2\alpha\cos^2\beta + \cosh^2\alpha\sin^2\beta} \\
&= c\sqrt{\sinh^2\alpha(1-\sin^2\beta) + \cosh^2\alpha\sin^2\beta} \\
&= c\sqrt{\sinh^2\alpha + \sin^2\beta(\cosh^2\alpha - \sinh^2\alpha)} \\
&= c\sqrt{\sinh^2\alpha + \sin^2\beta}\ ,
\end{aligned}$$

which we note is the same as the first scale factor, $h_\alpha = h_\beta$. We can now define $H(\alpha,\beta)$ to simplify our expression for the Laplacian,

$$H(\alpha,\beta) = \prod_j h_j = h_\alpha h_\beta h_\phi = h_\alpha^2 h_\phi = c^3(\sinh^2\alpha + \sin^2\beta)(\sinh\alpha\sin\beta)\ , \tag{9}$$

so that

$$\begin{aligned}
\nabla^2 = \frac{1}{H}\frac{\partial}{\partial q_i}\left[\frac{H}{h_i^2}\frac{\partial}{\partial q_i}\right] &= \frac{1}{H}\left\{\frac{\partial}{\partial\alpha}\left[\frac{H}{h_\alpha^2}\frac{\partial}{\partial\alpha}\right] + \frac{\partial}{\partial\beta}\left[\frac{H}{h_\beta^2}\frac{\partial}{\partial\beta}\right] + \frac{\partial}{\partial\phi}\left[\frac{H}{h_\phi^2}\frac{\partial}{\partial\phi}\right]\right\} \\
&= \frac{1}{H}\left\{\frac{\partial}{\partial\alpha}\left[h_\phi\frac{\partial}{\partial\alpha}\right] + \frac{\partial}{\partial\beta}\left[h_\phi\frac{\partial}{\partial\beta}\right] + \frac{\partial}{\partial\phi}\left[\frac{h_\alpha^2}{h_\phi}\frac{\partial}{\partial\phi}\right]\right\} \\
&= \frac{1}{H}\left\{\left[(\partial_\alpha h_\phi)\partial_\alpha + h_\phi\partial_\alpha^2\right] + \left[(\partial_\beta h_\phi)\partial_\beta + h_\phi\partial_\beta^2\right] + \frac{h_\alpha^2}{h_\phi}\partial_\phi^2\right\}\ ,
\end{aligned}$$

where ∂_k^i is the ith order partial derivative with respect to the kth coordinate. We can note that

$$\partial_\alpha h_\phi = c\cosh\alpha\sin\beta \tag{10}$$

$$\partial_\alpha h_\phi = c\sinh\alpha\cos\beta \tag{11}$$

$$h_\alpha^2/h_\phi = c\frac{\sinh^2\alpha + \sin^2\beta}{\sinh\alpha\sin\beta} = c(\sinh\alpha\csc\beta + \text{csch}\alpha\sin\beta)\ , \tag{12}$$

and thus the Laplacian operator is

$$\begin{aligned}
\nabla^2_{\{\alpha,\beta,\phi\}} = \frac{1}{c^2(\sinh^2\alpha+\sin^2\beta)(\sinh\alpha\sin\beta)}&\left\{(\cosh\alpha\sin\beta)\frac{\partial}{\partial\alpha} + (\sinh\alpha\sin\beta)\frac{\partial^2}{\partial\alpha^2}\right. \\
&\left. + (\sinh\alpha\cos\beta)\frac{\partial}{\partial\beta} + (\sinh\alpha\sin\beta)\frac{\partial^2}{\partial\beta^2} + (\sinh\alpha\csc\beta + \text{csch}\alpha\sin\beta)\frac{\partial^2}{\partial\phi^2}\right\}\ ,
\end{aligned} \tag{13}$$

which can be simplified by distributing the last term in the denominator of the overall factor,

$$\nabla^2_{\{\alpha,\beta,\phi\}} = \frac{1}{c^2(\sinh^2\alpha+\sin^2\beta)}\left\{\coth\alpha\frac{\partial}{\partial\alpha} + \frac{\partial^2}{\partial\alpha^2} + \cot\beta\frac{\partial}{\partial\beta} + \frac{\partial^2}{\partial\beta^2} + (\csc^2\beta + \text{csch}^2\alpha)\frac{\partial^2}{\partial\phi^2}\right\}\ , \tag{14}$$

which is the Laplacian operator in prolate spheroidal coordinates.

1.2 Solving Laplace's Equation.

We are interested in solving Laplaces equation, $\nabla^2_{\{\alpha,\beta,\phi\}} u = 0$. We will assume a separable form for u, with azimuthal symmetry, so that

$$u(\alpha,\beta) = C_\phi A(\alpha) B(\beta) \ , \tag{15}$$

where C_ϕ is some arbitrary constant that incorporates the azimuthal symmetry ($\Phi(\phi) = cst$). If we act the Laplacian operator (Equation 14)) on this ansatz, we find

$$\nabla^2 u = 0 = \frac{C_\phi}{c^2(\sinh^2\alpha + \sin^2\beta)} \left[\coth\alpha \frac{\mathrm{d}A}{\mathrm{d}\alpha} B + \frac{\mathrm{d}^2 A}{\mathrm{d}\alpha^2} B + \cot\beta \frac{\mathrm{d}B}{\mathrm{d}\beta} A + \frac{\mathrm{d}^2 B}{\mathrm{d}\beta^2} A + 0 \right] , \tag{16}$$

because there is no azimuthal dependence ($\partial_\phi = 0$). If we divide out the factor out front, then divide through by the quantity AB, we get the equation

$$-k^2 + k^2 = \left[\coth\alpha \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}\alpha} + \frac{1}{A}\frac{\mathrm{d}^2 A}{\mathrm{d}\alpha^2} + \cot\beta \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}\beta} + \frac{1}{B}\frac{\mathrm{d}^2 B}{\mathrm{d}\beta^2} \right] \tag{17}$$

where k is some real, arbitrary constant. This results in the two independent differential equations

$$k^2 A = \coth\alpha \frac{\mathrm{d}A}{\mathrm{d}\alpha} + \frac{\mathrm{d}^2 A}{\mathrm{d}\alpha^2} \tag{18}$$

$$-k^2 B = \cot\beta \frac{\mathrm{d}B}{\mathrm{d}\beta} + \frac{\mathrm{d}^2 B}{\mathrm{d}\beta^2} \ , \tag{19}$$

but can be simplified if we take derivatives with respect to $\cosh\alpha$ and $\cos\beta$. The differential operators with respect to α become

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\alpha} &= \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha}\frac{\mathrm{d}\cosh\alpha}{\mathrm{d}\alpha} = \sinh\alpha \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha} \\
\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\sinh\alpha \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha}\right] = \cosh\alpha \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha} + \sinh\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\cosh\alpha} \\
&= \cosh\alpha \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha} + \sinh\alpha \left[\sinh\alpha \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha}\right] \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha} \\
&= \cosh\alpha \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha} + \sinh^2\alpha \frac{\mathrm{d}^2}{\mathrm{d}\cosh\alpha^2} \ ,
\end{aligned}$$

and for β,

$$\frac{\mathrm{d}}{\mathrm{d}\beta} = \frac{\mathrm{d}}{\mathrm{d}\cos\beta}\frac{\mathrm{d}\cos\beta}{\mathrm{d}\beta} = -\sin\beta \frac{\mathrm{d}}{\mathrm{d}\cos\beta}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\beta^2} = -\cos\beta \frac{\mathrm{d}}{\mathrm{d}\cos\beta} - \sin\beta \left[-\sin\beta \frac{\mathrm{d}}{\mathrm{d}\cos\beta}\right] \frac{\mathrm{d}}{\mathrm{d}\cos\beta} = -\cos\beta \frac{\mathrm{d}}{\mathrm{d}\cos\beta} + \sin^2\beta \frac{\mathrm{d}^2}{\mathrm{d}\cos\beta^2} \ ,$$

so the differential equations become

$$0 = \cosh\alpha A + \sinh^2\alpha A'' + \coth\alpha \sinh\alpha A' - k^2 A \tag{20}$$

$$0 = -\cos\beta B' + \sin^2\beta B'' - \cot\beta \sin\beta B' + k^2 B \ , \tag{21}$$

where the primes denote derivatives with respect to $\cosh\alpha$ and $\cos\beta$. Upon simplification, and multiplying the equation for A by negative one, we see both equations are now in the form of Legendre's equation:

$$0 = -(\cosh^2\alpha - 1)A'' - 2\cosh\alpha A' + k^2 A \tag{22}$$

$$0 = (1 - \cos^2\beta)B'' - \cos\beta B' + k^2 B \ . \tag{23}$$

The first equation has $x = \cosh\alpha$ and $n(n+1) = k^2$, while the second has $x = \cos\beta$ and $n(n+1) = k^2$, so the solutions are Legendre polynomials of the same order n, with the respective argument. It is legitimate to set the arbitrary constant k^2 to the indicated value, so that the series solution will stay bounded at $\theta = 0, \pi$ [1]. We can represent the solution to Laplace's equation as a series of these solutions over the order of the Legendre polynomial. Incorporating the constant from the ϕ coordinate with the constant for the series, we see the solution can be written

$$u(\alpha, \beta, \phi) = \sum_{n=0}^{\infty} A_n P_n(\cosh\alpha) P_n(\cos\beta) \ , \tag{24}$$

which proves Equation 3.

[1] Arfken. Mathematical Methods for Physicists, 7 ed. Page 716.

2 Problem #2: Jackson 4.1 a,b.

Calculate the moments (up to quadrupolar order) of the two charge distributions shown in Figure 1 in both Cartesian and spherical coordinates. Also calculate the far-field potential to the same order.

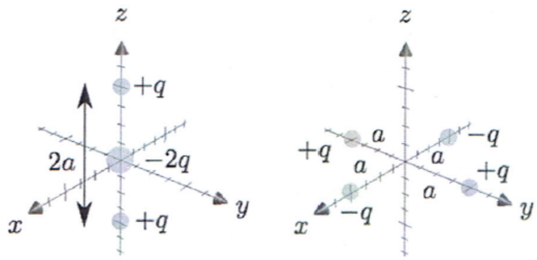

Figure 1: Charge distributions used in Problem #2.

The integrals for the dipole moment (Jackson equation 4.8) and the elements of the traceless quadrupole moment tensor (Jackson equation 4.9) of a discrete charge distribution become sums over each particle:

$$\mathbf{p} = \sum_{n=1}^{N} q_n \mathbf{r}_n \tag{25}$$

$$Q_{ij} = \sum_{n=1}^{N} q_n (3x_i x_j - r^2 \delta_{ij}) \ , \tag{26}$$

expressed in Cartesian coordinates. After these moments are calculated, they can be transformed to spherical coordinates by using Jackson Equations 4.4 through 4.6. Jackson equation 4.10 shows the form of the far-field potential in Cartesian coordinates, while Jackson equation 4.1 is the far field potential in spherical coordinates (in terms of the multipole moments).

2.1 First Distribution.

This distribution has charges q located at $(0, 0, \pm a)$ and a charge $-2q$ at the origin. We can immediately see that the monopole moment is zero because the total charge is zero. We note the central charge has does not contribute to the dipole or quadrupole moments because the value of its position coordinates are all zero. Therefore, the dipole moment is

$$\mathbf{p} = q\left[(a\hat{\mathbf{z}}) + (-a\hat{\mathbf{z}})\right] = 0 \ , \tag{27}$$

so this distribution only has quadrupole moments or higher. We can write the elements of the quadrupole moment tensor as

$$Q_{ij} = q\left[(3x_i^{(1)}x_j^{(1)} - a^2\delta_{ij}) + (3x_i^{(3)}x_j^{(3)} - a^2\delta_{ij})\right] = q\left[3(x_i^{(1)}x_j^{(1)} + x_i^{(3)}x_j^{(3)}) - 2a^2\delta_{ij}\right] \ , \tag{28}$$

where the superscripts are the particle index. We can note that $x_1^{(n)} = x_2^{(n)} = 0$, so all off-diagonal terms must vanish because they must have at least one $x_i = 0$, and the Kronecker delta is zero for these terms. Additionally, we note that $Q_{11} = Q_{22} = -2a^2$ because the x_i terms are all zero. The final terms is

$$Q_{33} = q[3(a^2 + a^2) - 2a^2] = 4a^2 \ , \tag{29}$$

so the quadrupole moment tensor is

$$\mathbf{Q}_{ij} = a^2 q \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} , \tag{30}$$

which is indeed traceless, as expected. Under a transformation to spherical coordinates we see

$$q_{00} = q_{11} = q_{10} = q_{1(-1)} = 0 , \tag{31}$$

due to the charge configuration having zero monopole and dipole moments. Additionally, the matrix elements $Q_{ij} = 0$ for $i \neq j$, and $Q_{11} = Q_{22}$ so we see

$$q_{22} = q_{21} = q_{2(-2)} = q_{2(-1)} = 0 . \tag{32}$$

Therefore, in spherical coordinates, this charge configuration only has one nonzero moment

$$q_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(4a^2 q) = \sqrt{\frac{5}{\pi}} a^2 q . \tag{33}$$

2.1.1 Far-Field Potential.

In Cartesian coordinates, the potential at large distances is given by

$$\Phi(\mathbf{r}) = \Phi(x, y, z) = \frac{1}{4\pi\epsilon_0} 2a^2 q \frac{z^2}{r^5} \tag{34}$$

while in spherical coordinates, the far field potential is

$$\Phi(\mathbf{r}) = \Phi(r, \theta, \phi) = \frac{1}{\epsilon_0}\frac{1}{5}\sqrt{\frac{5}{\pi}} a^2 q \frac{Y_{20}(\theta, \phi)}{r^3} \tag{35}$$

and such, these are valid for $|\mathbf{r}| > a$.

2.2 Second Distribution.

In this distribution, there is also a zero monopole moment because the total charge is zero. The dipole moment is also zero because each charge has a partner with the same magnitude at an opposite position, *i.e.*, $Q(\mathbf{r}) = Q(-\mathbf{r})$, so each term in the dipole moment sum cancels exactly with another. The elements of the quadrupole moment tensor are given by

$$Q_{ij} = q\left[3(x_i^{(2)} x_j^{(2)} + x_i^{(4)} x_j^{(4)} - x_i^{(1)} x_j^{(1)} - x_i^{(3)} x_j^{(3)}) - 2a^2\delta_{ij} + 2a^2\delta_{ij}\right] , \tag{36}$$

where the superscripts denote which particle's position to use. We note that each particle has a z coordinate of zero, so the off diagonal terms involving z are zero, as well as the 33 term because there is no Kronecker delta. The off diagonal elements Q_{12} and Q_{21} are also zero because each charge only has one nonzero component, so the product of two different components will always be zero. The remaining elements are

$$Q_{11} = q3(-a^2 - a^2) = -6a^2 q \tag{37}$$

$$Q_{22} = q3(a^2 + a^2) = 6a^2 q , \tag{38}$$

in Cartesian coordinates. Though explicit calculation of Q_{22} was unnecessary if we exploit the fact the quadrupole moment tensor is traceless, $Q_{11} = -Q_{22}$. After transforming to spherical coordinates, we see the multipole moments are

$$q_{00} = q_{11} = q_{10} = q_{21} = q_{20} = q_{2(-1)} = 0 \tag{39}$$

$$q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}(-2a^2q - 2a^2q) = \sqrt{\frac{5}{2\pi}}a^2q \tag{40}$$

$$q_{2(-2)} = (-1)^2 q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}(-2a^2q - 2a^2q) = \sqrt{\frac{5}{2\pi}}a^2q \ . \tag{41}$$

2.2.1 Far-Field Potential.

Using the appropriate equations from Jackson we find

$$\Phi(\mathbf{r}) = \Phi(x,y,z) = \frac{1}{8\pi\epsilon_0}\left[-2a^2q\frac{x^2}{r^5} + 2a^2q\frac{y^2}{r^5}\right] = \frac{1}{4\pi\epsilon_0}\frac{a^2q}{r^5}(y^2 - x^2) \tag{42}$$

$$\Phi(r,\theta,\phi) = \frac{1}{5\epsilon_0}\sqrt{\frac{5}{2\pi}}a^2q\frac{Y_{22}(\theta,\phi) + Y_{2(-2)}(\theta,\phi)}{r^3} \ , \tag{43}$$

which are valid for $|\mathbf{r}| > a$.

3 Problem #3: Jackson 4.7 a.

A localized charge distribution has a charge density

$$\rho(\mathbf{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta \ . \tag{44}$$

3.1 Multipole Expansion.

Make a multipole expansion of the potential due to this charge density and determine all the non-vanishing multipole moments.

Using Jackson equation 4.3, we can find all the multipole moments

$$q_{lm} = \frac{1}{64\pi} \int Y^*_{lm}(\theta,\phi) r^{l+2} e^{-r} \sin^2\theta (r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi) \tag{45}$$

$$= \frac{1}{32} \int_0^\infty r^{l+4} e^{-r} \mathrm{d}r \int_0^\pi Y^*_{lm}(\theta,\phi) \sin^3\theta \mathrm{d}\theta \ . \tag{46}$$

If we note the azimuthal symmetry of the charge distribution, it must be that $m = 0$ so that there is no ϕ dependence in the potential the charge distribution creates. Using this and writing the spherical harmonics in the form of Jackson equation 3.57, we see the multipole moments are of the form

$$q_{lm} = \frac{1}{32} \sqrt{\frac{2l+1}{4\pi}} \int_0^\infty r^{l+4} e^{-r} \mathrm{d}r \int_0^\pi P_l(\cos\theta) \sin\theta (1 - \cos^2\theta) \mathrm{d}\theta \ , \tag{47}$$

where $P_l(x)$ are the Legendre polynomials. We can investigate the polar integral under a change of variables $\cos\theta \to x$, and see that

$$I = \int_{-1}^1 P_l(x) \sin\theta (1-x^2)(-\sin\theta \mathrm{d}x) = \int_{-1}^1 P_l(x) \mathrm{d}x - \int_{-1}^1 x^2 P_l(x) \mathrm{d}x \ . \tag{48}$$

We should note that $P_0(x) = 1$, then we can exploit the orthogonality of Legendre polynomials to evaluate the integrals

$$I = \int_{-1}^1 P_0(x) P_l(x) \mathrm{d}x - \int_{-1}^1 x^2 P_0(x) P_l(x) \mathrm{d}x \tag{49}$$

$$= \frac{2\delta_{l0}}{2l+1} \mathrm{d}x - \int_{-1}^1 x^2 P_0(x) P_l(x) \mathrm{d}x \ . \tag{50}$$

Integrals of this form have solutions given by Jackson equation 3.32, which is nonzero only for $l = 0, 2$, so there are only two multipole moments. Therefore, this integral is

$$I = \frac{2}{2l+1} \left[\delta_{l0} - \frac{4}{75} \delta_{l2} - \frac{1}{3} \delta_{l0} \right] = \begin{cases} \frac{4}{3} & l = 0 \\ -\frac{4}{15} & l = 2 \end{cases} \ , \tag{51}$$

so the multipole moments are

$$q_{00} = \frac{1}{32} \frac{1}{\sqrt{4\pi}} \frac{4}{3} \int_0^\infty r^4 e^{-r} \mathrm{d}r = \frac{1}{\sqrt{4\pi}} \tag{52}$$

$$q_{20} = \frac{1}{32} \sqrt{\frac{5}{4\pi}} \frac{-8}{105} \int_0^\infty r^6 e^{-r} \mathrm{d}r = -6 \sqrt{\frac{5}{4\pi}} \ . \tag{53}$$

3.2 Far-Field Potential.

Write down the potential at large distances as a finite expansion in Legendre polynomials.

The multipole expansion of the potential due to the charge distribution, by Jackson equation 4.1, is

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\left[4\pi\frac{1}{\sqrt{4\pi}}\frac{Y_{00}(\theta,\phi)}{r} - \frac{4\pi}{5}6\sqrt{\frac{5}{4\pi}}\frac{Y_{20}(\theta,\phi)}{r^3}\right] \tag{54}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\sqrt{4\pi}\frac{Y_{00}(\theta,\phi)}{r} - 6\sqrt{\frac{4\pi}{5}}\frac{Y_{20}(\theta,\phi)}{r^3}\right] \tag{55}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{6}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\right] \tag{56}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{6}{r^3}P_2(\cos\theta)\right] \ , \tag{57}$$

which is the far-field potential.

4 Problem #4: Jackson 4.2.

By calculating the moments up to quadrupolar order, show that the charge distribution

$$\rho(\mathbf{r}) = -(\mathbf{p} \cdot \nabla)\delta(\mathbf{r}) \ , \tag{58}$$

represents an elementary dipole of moment $\mathbf{p}'$ at the origin. (This is most easily done in Cartesian coordinates).

In Cartesian coordinates, the charge distribution can be expressed as

$$\rho(x, y, z) = -\left[p'_x \frac{\partial}{\partial x} + p'_y \frac{\partial}{\partial y} + p'_z \frac{\partial}{\partial z}\right] \delta(x)\delta(y)\delta(z) \ , \tag{59}$$

where p'_i is the ith component of $\mathbf{p}'$. The relevant equations in Jackson are, equation 4.3 with $l = m = 0$ for the monopole moment, equation 4.8 for dipole moment, and equation 4.9 for the quadrupole moment tensor components. Note that all integration takes place over all space: $x_i \in (-\infty, \infty)$, so the limits of integration will be excluded, as they are implied.

4.1 Monopole Moment.

The monopole moment is given by

$$q_{00} = \int \rho(\mathbf{r}')Y_{00}(\theta', \phi')d\mathbf{r}' = -\frac{1}{\sqrt{4\pi}} \iiint \left[p'_x \frac{\partial}{\partial x'} + p'_y \frac{\partial}{\partial y'} + p'_z \frac{\partial}{\partial z'}\right] \delta(x')\delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z', \tag{60}$$

which if we distribute the three delta functions, we get three equivalent integrals, one for each Cartesian coordinate. Let us define the integral I_1 as

$$I_1 = \iiint \left[p'_x \frac{\partial}{\partial x'}\right] \delta(x')\delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \ , \tag{61}$$

and the integrals I_2 and I_3 follow. Then the monopole moment is given by $q_{00} = -(4\pi)^{-1/2}(I_1 + I_2 + I_3)$. Continuing the evaluation of the first integral is

$$I_1 = \iiint \left[p'_x \frac{\partial}{\partial x'}\delta(x')\right] \delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' = \int \left[p'_{x'} \frac{\partial}{\partial x'}\delta(x')\right] \mathrm{d}x' \ , \tag{62}$$

after integrating out the y' and z' delta functions, which over all space are unity. The integral can now be integrated by parts using $u = p'_x$ and $\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}x'}\delta(x')$,

$$I_1 = \left[p'_x\delta(x')\right]\Big|_{x'=-\infty}^{x'=\infty} - \int \frac{\mathrm{d}p'_x}{\mathrm{d}x'}\delta(x')\mathrm{d}x' = -\frac{\mathrm{d}p'_x}{\mathrm{d}x'}\Big|_{x'=0}, \tag{63}$$

where the boundary term is zero because it is evaluated at $\pm\infty$. The integrals I_2 and I_3 can be found similarly, which result in the monopole moment

$$q_{00} = -\frac{1}{\sqrt{4\pi}} \left[-\frac{\mathrm{d}p'_x}{\mathrm{d}x'} - \frac{\mathrm{d}p'_y}{\mathrm{d}y'} - \frac{\mathrm{d}p'_z}{\mathrm{d}z'}\right]\Big|_{x'=y'=z'=0} \ , \tag{64}$$

which is zero if the dipole moment $\mathbf{p}'$ does not depend on spatial coordinates. This is the case because $\mathbf{p}'$ is a constant vector.

4.2 Dipole Moment.

In Cartesian coordinates, the dipole moment of this charge distribution is

$$\mathbf{p} = -\iiint (x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}) \left[p'_x \frac{\partial}{\partial x'} + p'_y \frac{\partial}{\partial y'} + p'_z \frac{\partial}{\partial z'}\right] \delta(x')\delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \ , \tag{65}$$

again after distributing the delta functions, we get three similar integrals. The first of which is given by

$$I_1 = \iiint (x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}) \left[p'_x \frac{\partial}{\partial x'}\delta(x')\right] \delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \ , \tag{66}$$

and I_2 and I_3 follow similarly. The dipole moment is then given by $\mathbf{p} = -(I_1+I_2+I_3)$. The integral I_1 can be integrated by parts using $u = p'_x(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}})\delta(y')\delta(z')\mathrm{d}y'\mathrm{d}z'$, and $dv = \frac{\partial}{\partial x'}\delta(x')\mathrm{d}x'$. The boundary term is zero because the delta functions are only nonzero at $x' = y' = z' = 0$, and the boundary term is evaluated at $\pm\infty$. Now the integral is

$$\begin{aligned} I_1 &= -\iiint \frac{\mathrm{d}}{\mathrm{d}x}\left[p'_x x'\hat{\mathbf{x}} + p'_x y'\hat{\mathbf{y}} + p'_x z'\hat{\mathbf{z}}\right] \delta(x')\delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' & (67)\\ &= -\iiint \left[\left(\frac{\mathrm{d}p'_x}{\mathrm{d}x'}x' + p'_x\right)\hat{\mathbf{x}} + \frac{\mathrm{d}p'_x}{\mathrm{d}x'}y'\hat{\mathbf{y}} + \frac{\mathrm{d}p'_x}{\mathrm{d}x'}z'\hat{\mathbf{z}}\right] \delta(x')\delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \ . & (68) \end{aligned}$$

The integral over y' and z', due to the delta function, causes the integrand to be evaluated at $y' = z' = 0$, so the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ terms vanish. Finally, we see

$$I_1 = -\int \left(\frac{\mathrm{d}p'_x}{\mathrm{d}x'}x' + p'_x\right) \delta(x')\mathrm{d}x'\hat{\mathbf{x}} \ , \tag{69}$$

where the first term is zero due to the delta function, so $I_1 = -p'_x\hat{\mathbf{x}}$, and I_2 and I_3 follow similarly. We see the dipole moment is

$$\mathbf{p} = \left[p'_x\hat{\mathbf{x}} + p'_y\hat{\mathbf{y}} + p'_z\hat{\mathbf{z}}\right]\Big|_{\mathbf{r}=0} = \mathbf{p}' \ , \tag{70}$$

so the dipole moment of this charge configuration is the dipole moment $\mathbf{p}'$ at the origin.

4.3 Quadrupole Moment.

Inserting the charge distribution into Jackson equation 4.9, we see

$$Q_{ij} = -\iiint [3x'_i x'_j - (r')^2\delta_{ij}] \left[p'_x \frac{\partial}{\partial x'} + p'_y \frac{\partial}{\partial y'} + p'_z \frac{\partial}{\partial z'}\right] \delta(x')\delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \ , \tag{71}$$

where $(r')^2 = (x')^2 + (y')^2 + (z')^2$, and $x_i \in \{x, y, z\}$. Again, we can distribute the delta functions and get three similar integrals so that $Q_{ij} = -(I_1 + I_2 + I_3)$, the first of which is

$$I_1 = \iiint [3x'_i x'_j - (r')^2\delta_{ij}]\delta(y')\delta(z') \left[p'_x \frac{\partial}{\partial x'}\delta(x')\right] \mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \ , \tag{72}$$

which can be integrated by parts by using

$$\begin{aligned} u &= [3x'_i x'_j - (r')^2\delta_{ij}]\delta(y')\delta(z')p'_x\mathrm{d}y'\mathrm{d}z' & (73)\\ \mathrm{d}v &= \frac{\partial}{\partial x'}\delta(x')\mathrm{d}x' \ , & (74) \end{aligned}$$

from which we see

$$\mathrm{d}u = \frac{\partial}{\partial x'}\left[3x_i'x_j' - (r')^2\delta_{ij}\right]\mathrm{d}x'(p_x'\delta(y')\delta(z')\mathrm{d}y'\mathrm{d}z') \tag{75}$$

$$v = \delta(x')\ , \tag{76}$$

note p_x', p_y', p_z' are all constant. We can note the boundary term is zero because the delta functions are evaluated at $\pm\infty$, and are thusly zero. We now see our integral is

$$I_1 = -\iiint \delta(x')\frac{\partial}{\partial x'}\left[3x_i'x_j' - (r')^2\delta_{ij}\right]\mathrm{d}x'(p_x'\delta(y')\delta(z')\mathrm{d}y'\mathrm{d}z') \tag{77}$$

$$= -\iiint p_x'\left[3\frac{\partial x_i'}{\partial x'}x_j' + 3x_i'\frac{\partial x_j'}{\partial x'} - 2x'\delta_{ij}\right]\delta(x')\delta(y')\delta(z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\ , \tag{78}$$

which will always be zero when integrated over all space, for any i, j. This must be the case because the triple integral pulls out the integrand evaluated at the location the delta functions are nonzero. This occurs at $x' = y' = z' = 0$. For any Q_{ij} there is always a coordinate in each term that will be set to zero, so the integral will always be zero. The construction of I_1 is identical to that of I_2 and I_3 except with derivatives taken with respect to different coordinates. Therefore the result for I_1 will hold for the other two integrals, so if $I_1 = 0\ \forall\ \{i, j\}$, then $I_2 = I_3 = 0$. Using this result we see that

$$Q_{ij} = 0\ . \tag{79}$$

The charge distribution given by Equation 58, has no monopole or quadrupolar contributions to the potential. It's dipolar contribution is exactly that of a dipole of moment $\mathbf{p}'$ located at the origin. Therefore this charge distribution describes a perfect dipole of moment $\mathbf{p}'$ at the origin.