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Contents

1 Problem #1: Coaxial Conductors.

A cylindrical conductor of radius a is surrounded by a thin cylindrical conducting shell of radius b. The two conductors are coaxial, but carry equal and opposite currents I. Determine the magnetic field B due to this system at all points in space by integrating the differential form of Maxwell's equations. Check your results by using the integral form of these equations.

If we let the inner conductor carry current $+I$, and the outer carry ${}_I$, each distributed over their respective areas (in the $x - y$ plane) then, it is easy to see the current density can be expressed in cylindrical coordinates as

$$\mathbf{J}(\rho) = \hat{z}\left[\frac{I}{\pi a^2}\Theta(a-\rho) - \frac{I}{2\pi b}\delta(b-\rho)\right] \ , \tag{1}$$

where Θ is the Heaviside step function, which is one for $\rho < a$, and zero else, and δ is the Dirac delta function.

1.1 Differential Form.

The differential forms of Maxwell's equations that are relevant are

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \ . \tag{3}$$

Due to the infinite extent of the cylinder in the z direction, the fields cannot depend on the z coordinate, can cannot vary in the z direction:

$$B_z = 0 \quad \text{and} \quad \frac{\partial}{\partial z} f(\rho, \varphi, z) = 0 \ , \tag{4}$$

where f is any arbitrary function of the cylindrical coordinates. We can express the first relevant equation in cylindrical coordinates as

$$0 = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\varphi}{\partial \varphi} \tag{5}$$

$$0 = \left(B_\rho + \rho\frac{\partial B_\rho}{\partial \rho}\right) + \frac{\partial B_\varphi}{\partial \varphi} \tag{6}$$

$$\frac{\partial B_\varphi}{\partial \varphi} = -\left(B_\rho + \rho\frac{\partial B_\rho}{\partial \rho}\right) \ , \tag{7}$$

which if we require $B_\rho = 0$, we see that $B_\varphi \neq B_\varphi(\varphi)$ (which is required because otherwise the solution to an axially symmetric problem would not be axially symmetric), so it must be that the azimuthal component of the magnetic field is solely dependent on the radial coordinate. We can now expand the curl of the magnetic field, and noting that the $\hat{\rho}$ and $\hat{\varphi}$ components do not contribute because they only depend on derivatives with respect to z, and the z-component of the magnetic field. We can now equate the curl of the magnetic field with the current density

$$\frac{1}{\rho}\left(\frac{\partial(\rho B_\varphi)}{\partial \rho} - \frac{\partial B_\rho}{\partial \varphi}\right) = \mu_0 J_z \tag{8}$$

$$\frac{\partial(\rho B_\varphi)}{\partial \rho} = B_\varphi + \rho\frac{\partial B_\varphi}{\partial \rho} = \mu_0 \rho J_z \ , \tag{9}$$

and integrating the final expression

$$\int B_\varphi \mathrm{d}\rho + \int \rho \frac{\partial B_\varphi}{\partial \rho} \mathrm{d}\rho = \int \mu_0 \rho J_z \mathrm{d}\rho \ . \tag{10}$$

The center integral can be evaluated using integration by parts:

$$\int \rho \frac{\partial B_\varphi}{\partial \rho} \mathrm{d}\rho = \rho B_\varphi - \int (\mathrm{d}\rho) B_\varphi \ , \tag{11}$$

which when we insert into Equation 10 , we notice the integral term above exactly cancels with the first integral in Equation 10, so we get the result

$$\rho B_\varphi = \int \mu_0 \rho J_z \mathrm{d}\rho \quad \Rightarrow \quad B_\varphi = \frac{\mu_0}{\rho} \int \rho J_z \mathrm{d}\rho \ . \tag{12}$$

We can evaluate this integral to find the magnetic field in the three regions of the problem: $\rho < a$ (region 1), $a < \rho < b$ (region 2), and $\rho > b$ (region 3). To find the magnetic field in the first region, we integrate from the origin to an ρ in this region:

$$B_\varphi = \frac{\mu_0}{\rho} \int_0^r \rho J_z \mathrm{d}\rho = \frac{\mu_0}{\rho} \int_0^\rho \rho' \frac{I}{\pi a^2} \Theta(a - \rho') \mathrm{d}\rho' = \frac{\mu_0 I}{\pi a^2} \frac{1}{\rho} \int_0^\rho \rho' \mathrm{d}\rho' = \frac{\mu_0 I}{2\pi a^2} \rho \ . \tag{13}$$

In the second region:

$$B_\varphi = \frac{\mu_0}{\rho} \int_0^\rho \rho' \frac{I}{\pi a^2} \Theta(a - \rho') \mathrm{d}\rho' = \frac{\mu_0 I}{\pi a^2 \rho} \left[\int_0^a \rho' \Theta(a - \rho') \mathrm{d}\rho' + \int_a^\rho \rho' \Theta(a - \rho') \mathrm{d}\rho' \right] \tag{14}$$

$$= \frac{\mu_0 I}{\pi a^2} \frac{1}{\rho} \left(\frac{a^2}{2} \right) = \frac{\mu_0 I}{2\pi} \frac{1}{\rho} \ , \tag{15}$$

and in the third:

$$B_\varphi = \frac{\mu_0}{\rho} \int_0^\rho \left(\rho' \frac{I}{\pi a^2} \Theta(a - \rho') - \frac{I}{2\pi b} \delta(b - \rho) \right) \mathrm{d}\rho' \tag{16}$$

$$= \frac{\mu_0 I}{\pi a^2 \rho} \left[\int_0^a \rho' \Theta(a - \rho') \mathrm{d}\rho' + \int_a^b \rho' \Theta(a - \rho') \mathrm{d}\rho' \right] - \frac{\mu_0 I}{2\pi b \rho} \int_b^\rho \delta(b - \rho) \mathrm{d}\rho \tag{17}$$

$$= \frac{\mu_0 I}{\pi a^2} \frac{1}{\rho} \left(\frac{a^2}{2} \right) + \frac{\mu_0 I}{2\pi b} \frac{1}{\rho} (b) = 0 \ . \tag{18}$$

We now have the result for the magnetic field everywhere in space:

$$\mathbf{B}(\rho, \varphi, z) = B_\varphi \hat{\varphi} = \hat{\varphi} \frac{\mu_0 I}{2\pi} \begin{cases} \rho / a^2 & \rho < a \\ 1/\rho & a \leq \rho < b \\ 0 & \rho \geq b \end{cases} \ . \tag{19}$$

1.2 Integral Form.

To confirm the results of the previous section, we can use the integral form of Ampere's law,

$$\oint \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I_{\text{enc}} \ , \tag{20}$$

where I_{enc} is the current enclosed in the area bound by the path $\boldsymbol{\ell}$. The path $\boldsymbol{\ell}$ bounds a circle of radius r, and is in the azimuthal direction, so the above equation reduces to

$$B_\varphi(2\pi\rho) = \mu_0 I_{\text{enc}} \ . \tag{21}$$

The current on the inner conductor is distributed over the entire area (πa^2), so the current contained in a circle of radius $\rho < a$ is $(I/\pi a^2)(\pi\rho^2)$. Therefore in the three regions the magnetic field is

$$\text{Region 1}: \quad B_\varphi = \frac{\mu_0}{2\pi\rho} I_{\text{enc}} = \frac{\mu_0 I}{2\pi a^2}\rho \tag{22}$$

$$\text{Region 2}: \quad B_\varphi = \frac{\mu_0}{2\pi\rho} I_{\text{enc}} = \frac{\mu_0 I}{2\pi}\frac{1}{\rho} \tag{23}$$

$$\text{Region 3}: \quad B_\varphi = \frac{\mu_0}{2\pi\rho} I_{\text{enc}} = \frac{\mu_0}{2\pi\rho}(I - I) = 0 \ , \tag{24}$$

which confirm the previous results.

2 Problem #2: Magnetic Field inside Cylindrical Cavity.

Determine the magnetic field **B** in a cylindrical cavity inside an infinitely long cylindrical conductor. The radii of the cavity and the conductor are respectively a and b, and the distance between their parallel axes is d, with $b > a + d$. Assume that a current I is uniformly distributed across the conductor.

The current density of the conductor, if the current is distributed uniformly over the conductor's (cross-sectional) area, is

$$\mathbf{J} = (+\hat{z}) I \left(\pi b^2 - \pi a^2 \right)^{-1} = \frac{I}{\pi(b^2 - a^2)} \hat{\mathbf{z}} \; . \tag{25}$$

We can compare this problem to that of a cylindrical wire of radius a inside the larger conductor with a current that runs in the opposite direction. The cavity must have equal and opposite current density. Previously, we found the magnetic field inside a cylindrical conductor with cross-sectional area πa^2 and current I. By comparison, we can write the magnetic field inside a solid conductor of radius b, with the current density equivalent to that in our geometry (with the cavity):

$$B_\varphi = \frac{\mu_0 I}{2\pi(b^2 - a^2)} r \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_0 I}{2\pi(b^2 - a^2)} (\mathbf{r} \times \hat{\mathbf{z}}) \; , \tag{26}$$

which is valid at any point with $r < b$.

Now consider the magnetic field created by the conductor of radius a, a distance d from the origin. Let the vector from the origin to the axis of this conductor be **d**. Using the result from above, we can write the magnetic field inside this conductor as

$$\mathbf{B}' = \frac{\mu_0 (-I)}{2\pi(b^2 - a^2)} (|\mathbf{r} - \mathbf{d}| \times \hat{\mathbf{z}}) \; , \tag{27}$$

where **r** is a point inside the conductor, with respect to the origin (the axis of the conductor of radius b).

By the principle of superposition, we can write the magnetic field in the cavity as the sum of both magnetic fields found above:

$$\mathbf{B}_{\text{cavity}} = \frac{\mu_0 I}{2\pi(b^2 - a^2)} \left[(\mathbf{r} \times \hat{\mathbf{z}}) - (|\mathbf{r} - \mathbf{d}| \times \hat{\mathbf{z}}) \right] = \frac{\mu_0 I}{2\pi(b^2 - a^2)} \mathbf{d} \times \hat{\mathbf{z}} \tag{28}$$

$$= \frac{\mu_0 I}{2\pi(b^2 - a^2)} d \hat{\mathbf{r}} \times \hat{\mathbf{z}} = \frac{\mu_0 I d}{2\pi(b^2 - a^2)} \hat{\boldsymbol{\phi}} \; , \tag{29}$$

at points inside the cavity.

3 Problem #3: Axially Symmetric Magnetic Field.

3.1 Constant Flux through Cross-Section.

In previous classes, you have come across the concept of field lines, *i.e.*, lines of a vector field along which the vector is constant; for example, the magnetic field lines around a bar magnet. Consider then an axially symmetric magnetic field $\mathbf{B}$, and consider the tube generated by taking a magnetic field line and rotating it about the axis of symmetry. Show that the magnetic flux through any cross-section of this tube is constant.

If one field line of an axially symmetric magnetic field is rotated about the axis of symmetry, let this be the z axis, we get a volume of revolution centered around the z axis. Cross-sections of constant z will be circles of different radii, depending on $\mathbf{B}(a, z)$ where a is the [constant] radial position of the rotated field line. Consider the volume bound by the rotated field line and two of the cross-section circles. We can integrate the divergence of the magnetic field over this volume:

$$\int (\boldsymbol{\nabla} \cdot \mathbf{B}) \mathrm{d}V = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \ , \tag{30}$$

because $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. The magnetic flux through the tubular surface of this volume is zero (no radial component of the magnetic field), so the flux through the endcaps, S_1 and S_2 is

$$0 = \int_{S_1} \mathbf{B} \cdot A_1 \mathrm{d}(+\hat{\mathbf{z}}) + \int_{S_2} \mathbf{B} \cdot A_2 \mathrm{d}(-\hat{\mathbf{z}}) \ , \tag{31}$$

where A_1 and A_2 are the areas of the surfaces S_1 and S_2, respectively. From this, we can see

$$A_1 \int_{S_1} B_z \mathrm{d}z = A_2 \int_{S_2} B_z \mathrm{d}z \ , \tag{32}$$

which says the flux through is constant through surfaces formed by rotating an axially symmetric magnetic field line about its axis of symmetry.

3.2 Mangetic Field Line Equations.

Show that if a given magnetic field is axially symmetric and may be represented by a vector potential with cylindrical components $A_\theta(r, z),\ A_r = A_z = 0$, then the equations for the lines of magnetic field $\mathbf{B}$ is $rA_\theta(r, z) =$constant.

We found the flux through any surface made by rotating the field lines is constant, and this is given by

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell} \ , \tag{33}$$

by Stoke's theorem, where Φ is constant. Knowing that the magneitc field is axially symmetric, we may write the vector potential as

$$\mathbf{A} = A_\phi(r, z) \hat{\boldsymbol{\phi}} \ , \tag{34}$$

and the line element, in cylindrical coordinates is

$$\mathrm{d}\boldsymbol{\ell} = \mathrm{d}r \hat{\mathbf{r}} + r \mathrm{d}\phi \hat{\boldsymbol{\phi}} + \mathrm{d}z \hat{\mathbf{z}} \ . \tag{35}$$

If we insert these into the expression for magnetic flux, we find

$$\Phi = \int_0^{2\pi} A_\phi(r,z)(r\mathrm{d}\phi) = rA_\phi \int_0^{2\pi} \mathrm{d}\phi = 2\pi r A_\phi \ , \tag{36}$$

which if we rearrange, we get the result

$$rA_\phi = \frac{\Phi}{2\pi} = cst \ , \tag{37}$$

which is constant because the flux is a constant.

4 Problem #4: Infinite Sheet in Uniform Magnetic Field.

Consider a two dimensional (2D) sheet in the $x-y$ plane with a perpendicular magnetic field that gives rise to a vector potential

$$\mathbf{A} = \xi \frac{\hat{\mathbf{k}} \times \mathbf{r}}{r^2} \ , \tag{38}$$

where $\hat{\mathbf{k}}$ is the unit vector in the z direction, $\mathbf{r}$ is the position vector in the $x-y$ plane, and ξ is a constant.

4.1 Magnetic Field in the Plane.

What is the magnetic field in the $x-y$ plane?

If we express the vector potential in Cartesian coordinates, we obtain

$$\mathbf{A} = \xi \frac{1}{r^2} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ x & y & 0 \end{vmatrix} = \frac{\xi}{r^2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) = \frac{\xi}{r}(-\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}) = \frac{\xi}{r}\hat{\boldsymbol{\varphi}} \ , \tag{39}$$

where φ is the angle from the x axis, and is the azimuthal coordinate in cylindrical coordinates. The magnetic field is then given by

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times \frac{\xi}{r^2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) = \boldsymbol{\nabla} \times \xi r^{-2}\hat{\boldsymbol{\varphi}} \ , \tag{40}$$

which we will evaluate in Cartesian coordinates. We should note that since there is no dependence on z of $\mathbf{A}$, we can write $\frac{\partial \mathbf{A}}{\partial z} = 0$. This gives us

$$\mathbf{B} = \xi \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ A_x & A_y & 0 \end{vmatrix} = \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \xi\hat{\mathbf{z}}\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right) \ , \tag{41}$$

the first derivative is

$$\frac{\partial}{\partial x}\frac{x^2}{x^2+y^2} = \frac{1}{x^2+y^2} + x\frac{\partial}{\partial x}\frac{1}{x^2+y^2} = \frac{1}{x^2+y^2} + x\left(\frac{-2x}{(x^2+y^2)^2}\right) \ , \tag{42}$$

and the derivative for y must have the same form, just with y and x swapped. Summing these up we get

$$\mathbf{B} = \hat{\mathbf{z}}\xi\left(\frac{2}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2}\right) = \frac{2\xi}{r^2}\left(1 - \frac{x^2+y^2}{x^2+y^2}\right)\hat{\mathbf{z}} = 0 \ , \tag{43}$$

so the magnetic field in the $x-y$ plane is zero.

Let us note if we include z in $\mathbf{r}$ so $r^2 = x^2+y^2+z^2$, we get the same vector potential, but the first derivative in the curl is now

$$\frac{\partial}{\partial x}A_y = \frac{\xi}{x^2+y^2+z^2} - \frac{2\xi x^2}{(x^2+y^2+z^2)^2} \ , \tag{44}$$

so the magnetic field is just the difference of this and the derivative of A_y (which is the negative of this with x and y swapped):

$$\mathbf{B} = \frac{2\xi z^2}{(x^2+y^2+z^2)^2} \ , \tag{45}$$

which is still zero in the $x-y$ plane.

4.2 Quantum of Magnetic Flux.

In many materials that can be considered 2D, such as thin superconductors or 2D electron gases (2DEGs) such as those found in high mobility transistors (HEMTs), a perpendicular magnetic field can be thought of as penetrating the material as single magnetic field lines, each which pierces the 2D material at a single point, and each of which corresponds to a single quantum of magnetic flux $\Phi_0 = h/e$. By considering such a flux line at the origin, and integrating the expression for the vector potential given above around a closed path that encloses the origin, determine the constant ξ.

The magnetic flux through a surface S is given by

$$\Phi_B = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{l} \ , \tag{46}$$

by Stoke's theorem, where C is the contour bounding the surface S and $\mathbf{l}$ is the path around this contour. Consider the flux through a circle of radius R in the $x - y$ plane:

$$\Phi_B = \oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^{2\pi} A_\varphi \Big|_{r=R} (R\mathrm{d}\varphi) = \int_0^{2\pi} \frac{\xi}{R} R\mathrm{d}\varphi = 2\pi\xi \ . \tag{47}$$

If the circle of radius R is small enough, it will only contain a single quantum of magnetic flux, $\Phi_0 = h/e$, where e is the electron charge. If this is the case, we see

$$\frac{h}{e} = 2\pi\xi \quad \Rightarrow \quad \xi = \frac{h}{2\pi e} = \frac{\hbar}{e} \ . \tag{48}$$

5 Problem #5: Rotating Charged Sphere.

A sphere of radius R with a charge q uniformly distributed on its surface is rotating about a diameter at a constant angular velocity $\boldsymbol{\omega}$. Calculate the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ inside and outside the sphere.

Let us begin by defining the surface charge density $\sigma = q/(4\pi R^2)$, so that the current density is $\mathbf{J} = \sigma\mathbf{v}$ where $\mathbf{v}$ is the tangential velocity of a point on the surface. If we use the angular velocity we can write this as

$$\mathbf{J} = \sigma(\boldsymbol{\omega} \times \mathbf{r}') \ , \tag{49}$$

where, in Cartesian coordinates, a point on the surface is defined by the vector

$$\mathbf{r}' = R\sin\theta'\cos\varphi'\hat{\mathbf{x}} + R\sin\theta'\sin\varphi'\hat{\mathbf{y}} + R\cos\theta'\hat{\mathbf{z}} \ , \tag{50}$$

where θ' is the polar angle and φ' is the azimuthal angle. To define the angular velocity, we must now set the axes. We will consider a point in space $\mathbf{r}$, relative to the origin, and define the z axis to lie along this vector. We will then define the x axis such that the angular momentum will lie in the $x-z$ plane[1], see Figure 1a. Then, the angular momentum will make an angle ψ with the z axis:

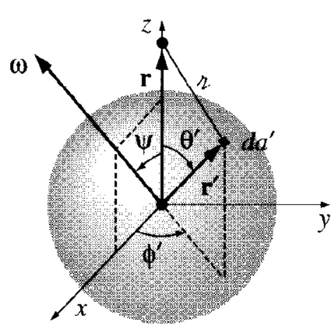
$$\boldsymbol{\omega} = \omega\sin\psi\hat{\mathbf{x}} + \omega\cos\psi\hat{\mathbf{z}} \ , \tag{51}$$

so the tangential velocity is

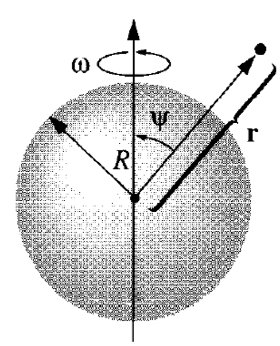
$$\begin{aligned}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\varphi' & R\sin\theta'\sin\varphi' & R\cos\theta'\hat{\mathbf{z}} \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(-\omega\cos\psi(R\sin\theta'\sin\varphi')\right) \\ &\quad + \hat{\mathbf{y}}\left(\omega\cos\psi(R\sin\theta'\cos\varphi') - \omega\sin\psi(R\cos\theta')\right) \\ &\quad + \hat{\mathbf{z}}\left(\omega\sin\psi(R\sin\theta'\sin\varphi')\right)\end{aligned}$$

which simplifies to

$$\mathbf{v} = \omega R\left[\hat{\mathbf{x}}\left(-\cos\psi\sin\theta'\sin\varphi)\right) + \hat{\mathbf{y}}\left(\cos\psi\sin\theta'\cos\varphi' - \sin\psi\cos\theta'\right) + \hat{\mathbf{z}}\left(\sin\psi\sin\theta'\sin\varphi'\right)\right] \tag{52}$$



(a) Coordinate system where the angular velocity makes an angle ψ with the z axis, in the $x-z$ plane.



(b) Coordinate system with the angular velocity aligned with the z axis.

[1]See Griffiths, Introduction to Electrodynamics, 3 ed. example 5.11.

The vector potential that arises from a current density[2] is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r} = \frac{\mu_0}{4\pi}\int \frac{\sigma(\boldsymbol{\omega}\times\mathbf{r}')}{\sqrt{r^2+R^2-2rR\cos\theta'}}\mathrm{d}^3r' \tag{53}$$

$$= \frac{\mu_0}{4\pi}\int_0^{\pi}\int_0^{2\pi} \frac{\sigma(\boldsymbol{\omega}\times\mathbf{r}')}{\sqrt{r^2+R^2-2rR\cos\theta'}}R^2\sin\theta'\mathrm{d}\theta'\mathrm{d}\varphi' \ , \tag{54}$$

because the volume integral, in this case is just over a surface at radius R, so the integration element is $R^2\sin\theta'\mathrm{d}\theta'\mathrm{d}\varphi'$ (because $\mathbf{r}'$ is defined as positions on the surface of the sphere). There are three terms in the tangential velocity that have dependence on φ', which when integrated over all space have factors of

$$\int_0^{2\pi}\sin\varphi'\mathrm{d}\varphi' = \int_0^{2\pi}\cos\varphi'\mathrm{d}\varphi' = 0 \ , \tag{55}$$

so there is only one non-zero term in the velocity. Additionally, the only r' dependence in the integrand is from the volume element, which integrates to R^2. We are left with only an integral over θ' for the vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0\omega R^3\sigma}{4\pi}(2\pi)\int_0^{\pi}\frac{-\sin\psi\cos\theta'\hat{\mathbf{y}}}{\sqrt{r^2+R^2-2rR\cos\theta'}}\sin\theta'\mathrm{d}\theta'\hat{\mathbf{y}} \ , \tag{56}$$

where the 2π came from the φ' integration. If we define $x=\cos\theta'$ ($\mathrm{d}x = -\sin\theta'\mathrm{d}\theta'$) simplifies to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 R^3\omega\sin\psi}{2}\left(\frac{q}{4\pi R^2}\right)\int_1^{-1}\frac{x\mathrm{d}x}{\sqrt{r^2+R^2-2rRx}}\hat{\mathbf{y}} \ , \tag{57}$$

letting Mathematica handle the integral, we obtain the result

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R(\omega q)\sin\psi}{8\pi}\left(\frac{-\left(r^2+rR+R^2\right)|r-R|+r^3+R^3}{3r^2R^2}\right) \ . \tag{58}$$

For regions inside and outside the sphere this is

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0\omega q\sin\psi}{8\pi}\begin{cases}\frac{2r}{3R^2} & r<R\\ \frac{2R}{3r^2} & r>R\end{cases} \tag{59}$$

in coordinates such that $\boldsymbol{\omega}$ lies in the $x-z$ plane, making an angle ψ with the z axis. We can collect a term $-\omega r\sin\psi$ and replace it with $\boldsymbol{\omega}\times\mathbf{r}$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 q}{12\pi}(\boldsymbol{\omega}\times\mathbf{r})\begin{cases}\dfrac{1}{R} & r<R\\ \dfrac{R^2}{r^3} & r>R\end{cases} \tag{60}$$

If we convert back to normal spherical coordinates (see Figure 1b), where $\boldsymbol{\omega}$ is aligned with $\hat{\mathbf{z}}$, we find that

$$\boldsymbol{\omega}\times\mathbf{r} = \omega\hat{\mathbf{z}}\times(r\sin\theta\cos\varphi\hat{\mathbf{x}} + r\sin\theta\sin\varphi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}) \tag{61}$$

$$= (-\omega r\sin\theta\sin\varphi)\hat{\mathbf{x}} + (\omega r\sin\theta\cos\varphi)\hat{\mathbf{y}} \tag{62}$$

$$= \omega r\sin\theta(-\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}) = \omega r\sin\theta\hat{\boldsymbol{\varphi}} \ . \tag{63}$$

[2]Jackson, Classical Electrodynamics, 3 ed. Equation 5.32

Using this, the vector potential everywhere in space, using spherical coordinates with $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, is

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 \omega q}{12\pi R} r \sin\theta \hat{\boldsymbol{\varphi}} & r < R \\ \dfrac{\mu_0 \omega q R^2}{12\pi} \dfrac{\sin\theta}{r^2} \hat{\boldsymbol{\varphi}} & r > R \end{cases} \tag{64}$$

and using $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ we can get the magnetic field. First consider the action of taking the curl:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial(A_\varphi \sin\theta)}{\partial\theta}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(-\frac{\partial(rA_\varphi)}{\partial r}\right)\hat{\boldsymbol{\theta}} \ , \tag{65}$$

and the derivatives are given by

$$\frac{\partial(A_\varphi \sin\theta)}{\partial\theta} = \sin\theta \frac{\partial A_\varphi}{\partial\theta} + A_\varphi \cos\theta \tag{66}$$

$$\frac{\partial(rA_\varphi)}{\partial r} = A_\varphi + r\frac{\partial A_\varphi}{\partial r} \ . \tag{67}$$

The magnetic is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial A_\varphi}{\partial\theta} + A_\varphi\cos\theta\right)\hat{\mathbf{r}} - \frac{1}{r}\left(A_\varphi + r\frac{\partial A_\varphi}{\partial r}\right)\hat{\boldsymbol{\theta}} \ . \tag{68}$$

Inside the sphere this is

$$\mathbf{B}_{\text{in}}(r,\theta,\varphi) = \frac{\mu_0\omega q}{12\pi R}\left[\frac{1}{r\sin\theta}\left(\sin\theta(r\cos\theta) + (r\sin\theta)\cos\theta\right)\hat{\mathbf{r}} - \frac{1}{r}\left((r\sin\theta) + r(\sin\theta)\right)\hat{\boldsymbol{\theta}}\right] \tag{69}$$

$$= \frac{\mu_0\omega q}{12\pi R}\left[2\cos\theta\hat{\mathbf{r}} - 2\sin\theta\hat{\boldsymbol{\theta}}\right] \tag{70}$$

$$= \frac{\mu_0\omega q}{6\pi R}\hat{\mathbf{z}} = \frac{\mu_0 q}{6\pi R}\boldsymbol{\omega} \ , \tag{71}$$

which is uniform everywhere inside. Outside the sphere, the magnetic field is

$$\mathbf{B}_{\text{out}}(r,\theta,\varphi) = \frac{\mu_0\omega q R^2}{12\pi}\left[\frac{1}{r\sin\theta}\left(\sin\theta\frac{\cos\theta}{r^2} + \frac{\sin\theta}{r^2}\cos\theta\right)\hat{\mathbf{r}} - \frac{1}{r}\left(\frac{\sin\theta}{r^2} + r\frac{-2\sin\theta}{r^3}\right)\hat{\boldsymbol{\theta}}\right] \tag{72}$$

$$= \frac{\mu_0\omega q R^2}{12\pi r^3}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right] \ . \tag{73}$$