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Contents

1 Dielectric Response of Quantum Electron Gas.

As we noted in class, the dielectric response of the electron gas that we have studied is rather a simple limit. Here we would like to derive a more sophisticated approximation (but still an approximation) taking into account the nature of the quantum gas. We still consider the electron gas to be free, *i.e.*, there are no external potentials aside from those imposed by the external electromagnetic fields.

Consider then a free electron gas in an external field that we take into account through an external potential $\phi_e(\mathbf{r}, t)$. If n_0 is the equilibrium density of the electrons, the external potential will lead to a perturbation δn in the density of the electron in response, so that the density is now $n = n_0 + \delta n$. This will lead in turn to an internal potential $\phi_i(\mathbf{r}, t)$ related to δn by Poisson's equation

$$\nabla^2 \phi_i(\mathbf{r}, t) = -\frac{e}{\epsilon_0} \delta n(\mathbf{r}, t) \ . \tag{1}$$

The total potential seen by the electrons is now $\phi = \phi_e + \phi_i$.

In absence of the external field, the Hamiltonian of the system is given by

$$H_0 \left|\mathbf{k}\right\rangle = -\frac{\hbar^2}{2m} \nabla^2 \left|\mathbf{k}\right\rangle = E(\mathbf{k}) \left|\mathbf{k}\right\rangle \ , \tag{2}$$

and the equilibrium state of the electron gas is described by the density operator ρ_0:

$$\rho_0 \left|\mathbf{k}\right\rangle = f_0(\mathbf{k}) \left|\mathbf{k}\right\rangle \ , \tag{3}$$

where $f_0(\mathbf{k})$ is the equilibrium Fermi distribution. Here $\left|\mathbf{k}\right\rangle \sim e^{i\kappa \cdot \mathbf{r}}$.

1.1 Heisenberg's Equation of Motion.

We now think of the external time-dependent potential as a perturbation on this Hamiltonian, *i.e.*, $H = H_0 + V$, where $V = -e\phi(\mathbf{r}, t)$, which in turn will lead to a change in density $\rho = \rho_0 + \delta\rho$. Using Heisenberg's equation of motion show that

$$i\hbar \left\langle \mathbf{k}' \right| \frac{\partial(\delta\rho)}{\partial t} \left|\mathbf{k}\right\rangle = (E(\mathbf{k}') - E(\mathbf{k})) \left\langle \mathbf{k}' \middle| \delta\rho \middle| \mathbf{k} \right\rangle - (f_0(\mathbf{k}') - f_0(\mathbf{k})) \left\langle \mathbf{k}' \middle| V \middle| \mathbf{k} \right\rangle \ . \tag{4}$$

The Heisenberg equation of motion for the time dependent operator A is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \frac{i}{\hbar}[H, A(t)] + \left(\frac{\partial A}{\partial t}\right)_H = \frac{1}{i\hbar}[A(t), H] + \left(\frac{\partial A}{\partial t}\right)_H \ , \tag{5}$$

where H is the Hamiltonian. However, since there is no parametric time dependence in the density operator ρ, we use the quantum Liouville equation[1]:

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H, \rho] \ . \tag{6}$$

[1] Also known as the von Neumann equation. Schwabl, Statistical Mechanics, equation 1.4.8.

Let us take the partial derivative of the operator ρ:

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho_0}{\partial t} + \frac{\partial (\delta\rho)}{\partial t} \quad \Rightarrow \quad \langle \mathbf{k}'|\frac{\partial (\delta\rho)}{\partial t}|\mathbf{k}\rangle = \langle \mathbf{k}'|\frac{\partial \rho}{\partial t}|\mathbf{k}\rangle - \langle \mathbf{k}'|\frac{\partial \rho_0}{\partial t}|\mathbf{k}\rangle \ . \tag{7}$$

If we use Equation 6, we now see

$$\langle \mathbf{k}'|\frac{\partial (\delta\rho)}{\partial t}|\mathbf{k}\rangle = \frac{1}{i\hbar}\langle \mathbf{k}'|[H,\rho]\mathbf{k}\rangle - \frac{1}{i\hbar}\langle \mathbf{k}'|[H_0,\rho_0]|\mathbf{k}\rangle \ , \tag{8}$$

note we have written the time derivative of the unperturbed density operator as the commutator of this with the unperturbed Hamiltonian. Since the unperturbed density does depend on the potential (is not changing with time) its time dependence would be totally due to the time dependence of the unperturbed Hamiltonian (spoiler: there is none). It is also evident that because the unperturbed Hamiltonian H_0 and the unperturbed density operator ρ_0 share common eigenfunctions, we have that

$$[H_0, \rho_0] = 0 \ , \tag{9}$$

using this, and the fact that neither operator is explicitly time-dependent, from Heisenberg's equation of motion, we see

$$\frac{\partial \rho_0}{\partial t} = 0 \ , \tag{10}$$

thus

$$i\hbar \langle \mathbf{k}'|\frac{\partial (\delta\rho)}{\partial t}|\mathbf{k}\rangle = \langle \mathbf{k}'|[H,\rho]\mathbf{k}\rangle \ . \tag{11}$$

Let us now consider the commutator of the perturbed Hamiltonian and the perturbed density:

$$[H,\rho] = [H_0 + V, \rho_0 + \delta\rho] = [H_0,\rho_0] + [H_0,\delta\rho] + [V,\rho_0] + [V,\delta\rho] \ , \tag{12}$$

note the first commutator, we have shown, is zero. Additionally, we may neglect the last commutator, because it is second-order in the perturbation: V is the time-dependent perturbation to the Hamiltonian, and $\delta\rho$ is the resultant perturbation to the density function. Taking the bra- and ket- with $\mathbf{k}'$ and $\mathbf{k}$, respectively, yields

$$\begin{aligned}
\langle \mathbf{k}'|[H,\rho]|\mathbf{k}\rangle &= \langle \mathbf{k}'|[H_0,\delta\rho]|\mathbf{k}\rangle + \langle \mathbf{k}'|[V,\rho_0]|\mathbf{k}\rangle &(13)\\
&= \langle \mathbf{k}'|H_0\delta\rho|\mathbf{k}\rangle - \langle \mathbf{k}'|\delta\rho H_0|\mathbf{k}\rangle + \langle \mathbf{k}'|V\rho_0|\mathbf{k}\rangle - \langle \mathbf{k}'|\rho_0 V|\mathbf{k}\rangle &(14)\\
&= (E(\mathbf{k}') - E(\mathbf{k}))\langle \mathbf{k}'|\delta\rho|\mathbf{k}\rangle + (f_0(\mathbf{k}) - f_0(\mathbf{k}'))\langle \mathbf{k}'|V|\mathbf{k}\rangle &(15)\\
&= (E(\mathbf{k}') - E(\mathbf{k}))\langle \mathbf{k}'|\delta\rho|\mathbf{k}\rangle - (f_0(\mathbf{k}') - f_0(\mathbf{k}))\langle \mathbf{k}'|V|\mathbf{k}\rangle \ , &(16)
\end{aligned}$$

and, using Equation 11 we obtain the result

$$i\hbar \langle \mathbf{k}'|\frac{\partial (\delta\rho)}{\partial t}|\mathbf{k}\rangle = (E(\mathbf{k}') - E(\mathbf{k}))\langle \mathbf{k}'|\delta\rho|\mathbf{k}\rangle - (f_0(\mathbf{k}') - f_0(\mathbf{k}))\langle \mathbf{k}'|V|\mathbf{k}\rangle \ . \tag{17}$$

1.2 Dielectric Constant - The Lindhard Formula.

The quantity $\langle \mathbf{k}'|V|\mathbf{k}\rangle$ can be defined as the Fourier transform of the perturbing potential $V(r,t)$

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = \frac{\mathbb{I}}{\mathcal{V}}\int (\mathrm{d}^3 r) e^{-i\mathbf{k}'\cdot\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{r}} V(r,t) = V_{\mathbf{q}}(t)/\mathcal{V} \ , \tag{18}$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ and $\mathcal{V}$ is the volume. We assume the time dependence of $V_e(r,t)$ is given by $\sim e^{-i\omega t}e^{\gamma t}$. The last exponential factor involving $\gamma > 0$ is an artifice to adiabatically turn on the external perturbation from $t = -\infty$ to $t = 0$; later, we will take the limit as $\gamma \to 0$. Assuming that the time dependence of all the other potentials and the response $\delta\rho$ is of the same form, and noting that

$$\delta n(\mathbf{r}, t) = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \delta\rho_{\mathbf{q}}(t) \ , \tag{19}$$

where

$$\delta\rho_{\mathbf{q}}(t) = \sum_{\mathbf{k}} \langle \mathbf{q}+\mathbf{k}|\delta\rho(t)|\mathbf{k}\rangle \tag{20}$$

show that the dielectric constant (really the relative permittivity), defined as the ratio ϕ_e/ϕ is given by the Lindhard formula

$$\epsilon(\mathbf{q}, \omega) = 1 - \lim_{\gamma\to 0} \frac{e^2}{\mathcal{V}\epsilon_0 q^2} \sum_{\mathbf{k}} \frac{f_0(\mathbf{k}+\mathbf{q}) - f_0(\mathbf{k})}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \hbar\omega - i\hbar\gamma} \ . \tag{21}$$

We are interested in determining the ratio

$$\frac{\phi_e}{\phi} = 1 - \frac{\phi_i}{\phi} \ , \tag{22}$$

where the potential energy is given by $V = -e\phi$. The internal potential ϕ_i satisfies the Poisson equation given in Equation 1, and has a discrete Fourier transform

$$\phi_i(\mathbf{r}) = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \phi_i(\mathbf{q}) \ . \tag{23}$$

We may write the potential energy, given above, in $\mathbf{q}$ space:

$$\frac{1}{\phi(\mathbf{q})} = -\frac{e}{V_{\mathbf{q}}} \ , \tag{24}$$

after Fourier transforming. If we insert the internal potential using the definition given by the Fourier transform into Equation 1, we obtain the relation

$$\nabla^2_{\mathbf{r}} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \phi_i(\mathbf{q}) = -\frac{e}{\epsilon_0} \delta n(\mathbf{r}, t) \ , \tag{25}$$

we may carry out the derivative over $\mathbf{r}$, and insert Equations 19 and 20, to yield

$$-\sum_{\mathbf{q}} |\mathbf{q}|^2 e^{i\mathbf{q}\cdot\mathbf{r}} \phi_i(\mathbf{q}) = -\frac{e}{\epsilon_0} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \sum_{\mathbf{k}} \langle \mathbf{q}+\mathbf{k}|\delta\rho(t)|\mathbf{k}\rangle \ . \tag{26}$$

Using this result, we have an expression for the internal potential in $\mathbf{q}$ space:

$$\phi_i(\mathbf{q}) = \frac{e}{\epsilon_0 |\mathbf{q}|^2} \sum_{\mathbf{k}} \langle \mathbf{q}+\mathbf{k}|\delta\rho(t)|\mathbf{k}\rangle \ . \tag{27}$$

Given the time dependence of the response $\delta\rho$ is exponential ($\sim e^{-i\omega t} e^{\gamma t}$), we may write

$$\langle \mathbf{k}+\mathbf{q}| \frac{\partial(\delta\rho)}{\partial t} |\mathbf{k}\rangle = \langle \mathbf{k}+\mathbf{q}|(-i\omega+\gamma)\delta\rho|\mathbf{k}\rangle = (-i\omega+\gamma) \langle \mathbf{k}+\mathbf{q}|\delta\rho|\mathbf{k}\rangle \ . \tag{28}$$

Using the identity given in Equation 4 to rewrite the left-hand side, and with a bit of rearranging, we find

$$(f_0(\mathbf{k}+\mathbf{q}) - f_0(\mathbf{k}))\,\langle \mathbf{k}+\mathbf{q}|V|\mathbf{k}\rangle = (E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}))\,\langle \mathbf{k}+\mathbf{q}|\delta\rho|\mathbf{k}\rangle - i\hbar(-i\omega+\gamma)\,\langle \mathbf{k}+\mathbf{q}|\delta\rho|\mathbf{k}\rangle \quad (29)$$

$$\frac{f_0(\mathbf{k}+\mathbf{q}) - f_0(\mathbf{k})}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \hbar\omega - i\hbar\gamma}\,\langle \mathbf{k}+\mathbf{q}|V|\mathbf{k}\rangle = \langle \mathbf{k}+\mathbf{q}|\delta\rho|\mathbf{k}\rangle \quad (30)$$

Let us insert Equation 30 into Equation 27, and combine with Equation 24 to rewrite Equation 22 as

$$\frac{\phi_e}{\phi} = 1 + \frac{e^2}{\epsilon_0|\mathbf{q}|^2}\sum_{\mathbf{k}} \frac{f_0(\mathbf{k}+\mathbf{q}) - f_0(\mathbf{k})}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \hbar\omega - i\hbar\gamma}\frac{\langle \mathbf{k}+\mathbf{q}|V|\mathbf{k}\rangle}{V_{\mathbf{q}}}\ , \quad (31)$$

using Equation 18, we see the final factor is the inverse volume. Finally, we take the limit as $\gamma \to 0$, to obtain the Lindhard formula for the dielectric constant (relative permittivity):

$$\epsilon(\mathbf{q},\omega) = 1 + \lim_{\gamma\to 0}\frac{e^2}{\epsilon_0\mathcal{V}|\mathbf{q}|^2}\sum_{\mathbf{k}} \frac{f_0(\mathbf{k}+\mathbf{q}) - f_0(\mathbf{k})}{E(\mathbf{k}+\mathbf{q}) - E(\mathbf{k}) - \hbar\omega - i\hbar\gamma}\ . \quad (32)$$

2 Rectangular Waveguide.

Find the nature of the TE and TM waves that can propagate in a rectangular waveguide with perfectly conducting walls and sides a and b. Determine the corresponding dispersion relations and the field configurations, *i.e.*, the dependence of the field components on the coordinates.

The longitudinal component of the field inside the waveguide is given by $\psi e^{\pm ikz}$, where ψ is E_z for TM waves and H_z for TE waves. This scalar function satisfies

$$(\nabla_t^2 + \gamma^2)\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2 \right) \psi = 0 \ , \tag{33}$$

where $\gamma^2 = \mu\epsilon\omega^2 - k^2$. If we assume a separable solution $\psi = X(x)Y(y)$, and define $\gamma^2 = k_x^2 + k_y^2$, we obtain the two equations

$$\frac{\partial^2 X(x)}{\partial x^2} = -k_x^2 X(x) \tag{34}$$

$$\frac{\partial^2 Y(y)}{\partial y^2} = -k_y^2 Y(y) \ , \tag{35}$$

which has solutions

$$X(x) = A_x \cos(k_x x) + B_x \sin(k_x x) \tag{36}$$

$$Y(y) = A_y \cos(k_y y) + B_y \sin(k_y y) \ \ , \tag{37}$$

where the constants A_i, B_i are determined by the appropriate boundary conditions for each supported mode.

2.1 Transverse Electric (TE) Mode.

The boundary conditions for a TE mode are

$$\left.\frac{\partial \psi}{\partial n}\right|_S = 0 \ , \tag{38}$$

where n is the normal direction to the surface S. For a rectangular waveguide of side lengths a and b, we have the following conditions:

$$0 = \left.\frac{\partial \psi}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi}{\partial x}\right|_{x=a} = \left.\frac{\partial \psi}{\partial y}\right|_{y=0} = \left.\frac{\partial \psi}{\partial y}\right|_{y=b} \ . \tag{39}$$

For a TE wave ψ corresponds to the magnetic field in the axial direction, the derivatives of which are given by

$$X'(x) = -A_x k_x \sin(k_x x) + B_x k_x \cos(k_x x) \tag{40}$$

$$Y'(y) = -A_y k_y \sin(k_y y) + B_y k_y \cos(k_y y) \ \ . \tag{41}$$

We now impose the boundary conditions:

$$X'(0) = 0 = -A_x k_x \sin(0) + B_x k_x \cos(0) \quad \Rightarrow \quad B_x = 0 \tag{42}$$

$$Y'(0) = 0 = -A_y k_y \sin(0) + B_y k_y \cos(0) \quad \Rightarrow \quad B_y = 0 \tag{43}$$

$$X'(a) = 0 = -A_x k_x \sin(k_x a) + B_x k_x \cos(k_x a) \quad \Rightarrow \quad k_x a = n_x \pi \tag{44}$$

$$Y'(b) = 0 = -A_y k_y \sin(k_y b) + B_y k_y \cos(k_y b) \quad \Rightarrow \quad B_y b = n_y \pi \ , \tag{45}$$

where $\{n_x, n_y\} \in \mathbb{Z}$, but because $\cos(x) = \cos(-x)$, we may restrict these parameters to $\{n_x, n_y\} \in \mathbb{N}$. Using this, we see the axial magnetic field is

$$H_z(x,y,z;t) = H_0 \cos\left(\frac{n_x \pi}{a} x\right) \cos\left(\frac{n_y \pi}{b} y\right) e^{ikz - i\omega t} \ , \tag{46}$$

with

$$\gamma^2_{n_x n_y} = \pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right) \ . \tag{47}$$

Using Jackson equation 8.38, we may define a cutoff frequency

$$\omega_{n_x n_y} = \frac{\pi}{\sqrt{\mu\epsilon}} \sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}} \ . \tag{48}$$

Let us note the rectangular waveguide cannot support a $n_x = n_y = 0$ mode because there will be no wave. If we assume $a > b$, the lowest cutoff frequency (corresponding to the dominant mode) is given by

$$\omega_{10} = \frac{\pi}{a\sqrt{\mu\epsilon}} \ . \tag{49}$$

The dispersion relation for TE waves can be determined by Jackson equation 8.37:

$$k^2_{n_x n_y} = \mu\epsilon\omega^2 - \pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right) \ , \tag{50}$$

which now replaces k in Equation 46. Using the expression for the transverse components of the magnetic field (Jackson equation 8.33), we see

$$\mathbf{H}_t = \pm \frac{ik_{n_x n_y}}{\pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right)} \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} \right) H_0 \cos\left(\frac{n_x \pi}{a} x\right) \cos\left(\frac{n_y \pi}{b} y\right) \tag{51}$$

$$= \mp H_0 \frac{ik_{n_x n_y}}{\pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right)} \left(\frac{\pi n_x}{a} \sin\left(\frac{n_x \pi}{a} x\right) \cos\left(\frac{n_y \pi}{b} y\right) \hat{\mathbf{x}} + \frac{\pi n_y}{b} \cos\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{b} y\right) \hat{\mathbf{y}} \right) \tag{52}$$

(not including the implicit carrier wave $\exp[ik_{n_x n_y} z - i\omega t]$) with Jackson equation 8.31 the transverse components of the electric field can be found:

$$\hat{\mathbf{z}} \times \mathbf{E}_t = \hat{\mathbf{z}} \times (E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}}) = E_x \hat{\mathbf{y}} - E_y \hat{\mathbf{x}} \ , \tag{53}$$

so

$$E_x = \mp Z(\hat{\mathbf{y}} \cdot \mathbf{H}_t) = H_0 \frac{i\mu\omega n_y}{\pi b \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right)} \cos\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{b} y\right) \tag{54}$$

$$E_y = \pm Z(\hat{\mathbf{x}} \cdot \mathbf{H}_t) = -H_0 \frac{i\mu\omega n_x}{\pi a \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right)} \sin\left(\frac{n_x \pi}{a} x\right) \cos\left(\frac{n_y \pi}{b} y\right) \ , \tag{55}$$

where $Z_{n_x n_y} = \mu\omega / k_{n_x n_y}$, again not including the implicit carrier wave $\exp[ik_{n_x n_y} z - i\omega t]$.

2.2 Transverse Magnetic (TM) Mode.

The boundary conditions for a TM mode are

$$\psi\big|_S = 0 \ , \tag{56}$$

where n is the normal direction to the surface S. For a rectangular waveguide of side lengths a and b, we have the following conditions:

$$0 = \psi(0, y) = \psi(a, y) = \psi(x, 0) = \psi(x, b) \ , \tag{57}$$

from which Equations 36 and 37 yield

$$X(0) = 0 = A_x \cos(0) + B_x \sin(0) \quad \Rightarrow \quad A_x = 0 \tag{58}$$

$$Y(0) = 0 = A_y \cos(0) + B_y \sin(0) \quad \Rightarrow \quad A_y = 0 \tag{59}$$

$$X(a) = 0 = B_x \sin(k_x a) \quad \Rightarrow \quad k_x a = n_x \pi \tag{60}$$

$$Y(b) = 0 = B_y \sin(k_y b) \quad \Rightarrow \quad B_y b = n_y \pi \ , \tag{61}$$

where $\{n_x, n_y\} \in \mathbb{Z}$. Using this, we see the axial magnetic field is

$$E_z(x, y, z; t) = E_0 \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{b} y\right) e^{ikz - i\omega t} \ , \tag{62}$$

with

$$\gamma^2_{n_x n_y} = \pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right) \ . \tag{63}$$

Using Jackson equation 8.38, we may define a cutoff frequency

$$\omega_{n_x n_y} = \frac{\pi}{\sqrt{\mu\epsilon}} \sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}} \ . \tag{64}$$

Let us note the rectangular waveguide cannot support a $n_x = n_y = 0$ mode because there will be no wave. If we assume $a > b$, the lowest cutoff frequency is given by

$$\omega_{10} = \frac{\pi}{a\sqrt{\mu\epsilon}} \ . \tag{65}$$

The dispersion relation for TM waves can be determined by Jackson equation 8.37:

$$k^2_{n_x n_y} = \mu\epsilon\omega^2 - \pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right) \ , \tag{66}$$

which now replaces k in Equation 62. Using the expression for the transverse components of the electric field (Jackson equation 8.33), we see

$$\mathbf{E}_t = \pm \frac{ik_{n_x n_y}}{\pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)} \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}\right) E_0 \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{b} y\right) \tag{67}$$

$$= \pm E_0 \frac{ik_{n_x n_y}}{\pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)} \left(\frac{\pi n_x}{a} \cos\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{b} y\right) \hat{\mathbf{x}} + \frac{\pi n_y}{b} \sin\left(\frac{n_x \pi}{a} x\right) \cos\left(\frac{n_y \pi}{b} y\right) \hat{\mathbf{y}}\right) \tag{68}$$

(not including the implicit carrier wave $\exp[ik_{n_xn_y}z - i\omega t]$) with Jackson equation 8.31 the transverse components of the magnetic field can be found:

$$\mathbf{H}_t = H_x\hat{\mathbf{x}} + H_y\hat{\mathbf{y}} = \frac{1}{Z}\hat{\mathbf{z}} \times \mathbf{E}_t = \frac{1}{Z}\hat{\mathbf{z}} \times (E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}) = \frac{1}{Z}(E_x\hat{\mathbf{y}} - E_y\hat{\mathbf{x}}) \ , \tag{69}$$

so

$$H_x = E_0 \frac{i\left[\mu\epsilon\omega^2 - \pi^2\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)\right] n_y}{\pi\epsilon\omega b\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)} \sin\left(\frac{n_x\pi}{a}x\right)\cos\left(\frac{n_y\pi}{b}y\right) \tag{70}$$

$$H_y = -E_0 \frac{i\left[\mu\epsilon\omega^2 - \pi^2\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)\right] n_x}{\pi a\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)} \cos\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{b}y\right) \ , \tag{71}$$

where $Z_{n_xn_y} = k_{n_xn_y}/\epsilon\omega$, again not including the implicit carrier wave $\exp[ik_{n_xn_y}z - i\omega t]$.

3 Circular Waveguide.

Repeat the problem above for a cylindrical waveguide of circular cross-section of radius a.

The longitudinal component of the electric or magnetic field (for TM and TE modes, respectively) is given by Jackson equation 8.34, in cylindrical coordinates:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\gamma^2\right)\psi=0\ , \tag{72}$$

where the longitudinal component is $\psi e^{\pm ikz}$, with $\gamma^2=\mu\epsilon\omega^2-k^2$. If we assume a separable solution of the form $\psi(\rho,\phi)=R(\rho)\Theta(\phi)$, the equation above becomes

$$\frac{1}{R}\rho\left(\frac{\mathrm{d}R}{\mathrm{d}\rho}+\rho\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}\right)+\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\phi^2}+\gamma^2\rho^2=0\ , \tag{73}$$

after dividing by $R(\rho)\Theta(\phi)$ and multiplying through by ρ^2. We have now separated the equation into terms of only one coordinate, which implies

$$-l^2\Theta=\frac{\mathrm{d}^2\Theta}{\mathrm{d}\phi^2} \tag{74}$$

$$l^2R=\rho\left(\frac{\mathrm{d}R}{\mathrm{d}\rho}+\rho\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}\right)+\gamma^2\rho^2R\ , \tag{75}$$

where l^2 is some constant. The first of the above equations has harmonic solutions

$$\Theta(\phi)=A\cos(l\phi)+B\cos(l\phi)\ . \tag{76}$$

The second can be rearranged to Bessel's equation:

$$0=\rho^2\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}+(\gamma^2\rho^2-l^2)R\ , \tag{77}$$

which has solutions

$$R(\rho)\sim J_l(\gamma\rho)\ , \tag{78}$$

where we have ignored the Bessel functions of the second kind because they diverge at the origin. Therefore the longitudinal waves have the form

$$\psi(\rho,\phi)=J_l(\gamma\rho)\left(A\sin(l\phi)+B\cos(l\phi)\right)\ . \tag{79}$$

We can immediately constrain the constant l for both TE and TM waves, by noting the behavior of $\Theta(\phi)$ is qualitatively the same as $\Theta'(\phi)$. Consider the function

$$f(x)=C_1\sin(lx)+C_2\cos(lx)\ , \tag{80}$$

which must obey periodicity:

$$f(x)=f(x+2\pi)=C_1\sin(lx+2\pi l)+C_2\cos(lx+2\pi l)\ . \tag{81}$$

For this to be satisfied, it must be that l is an integer, this corresponds to our angular solution: there can be no discontinuities in the field as we traverse around the axis of the circular waveguide, *i.e.*, the field at (ρ,ϕ,z) must equal the field at $(\rho,\phi+2\pi,z)$, which corresponds to the same physical location inside the waveguide.

3.1 Transverse Electric (TE) Mode.

The transverse electric mode has $E_z = 0$, and the axial magnetic field is

$$H_z = J_l(\gamma\rho)\left(A\sin(l\phi) + B\cos(l\phi)\right)e^{ikz} \ . \tag{82}$$

The boundary condition for $\psi = H_z e^{-ikz}$ in a TE mode in a circular waveguide of radius a is

$$\left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=a} = 0 \ , \tag{83}$$

which gives us the condition[2]

$$\left.\frac{\partial}{\partial\rho}J_l(\gamma\rho)\right|_{\rho=a} = 0 \quad \Rightarrow \quad \gamma a = \chi'_{nl} \quad \Rightarrow \quad \gamma_{nl} = \frac{\chi'_{nl}}{a}, \tag{84}$$

where χ'_{nl} is the nth zero of the first derivative of the lth Bessel function of the first kind, see the table Jackson, page 370. The dispersion relation is given by Jackson equation 8.37:

$$k_{nl}^2 = \mu\epsilon\frac{\chi'^2_{nl}}{a^2} \ . \tag{85}$$

Using Jackson equation 8.33 we can write the transverse components of the magnetic field as

$$\mathbf{H}_t = \pm\frac{ik_{nl}a^2}{\chi'^2_{nl}}\left(\left(A\sin(l\phi) + B\cos(l\phi)\right)\frac{\partial}{\partial\rho}J_l(\frac{\chi_{nl}}{a}\rho)\hat{\boldsymbol{\rho}} + \frac{J_l(\frac{\chi_{nl}}{a}\rho)}{\rho}\frac{\partial}{\partial\phi}\left(A\sin(l\phi) + B\cos(l\phi)\right)\hat{\boldsymbol{\phi}}\right)$$

Jackson equation 8.31 gives the relation

$$\pm Z(H_\rho\hat{\boldsymbol{\rho}} + H_\phi\hat{\boldsymbol{\phi}}) = E_\rho\hat{\boldsymbol{\phi}} - E_\phi\hat{\boldsymbol{\rho}} \ , \tag{86}$$

so

$$E_\rho = \pm Z(\hat{\boldsymbol{\phi}}\cdot\mathbf{H}_t) = \frac{i\mu\omega a^2}{\chi'^2_{nl}}\frac{J_l(\frac{\chi_{nl}}{a}\rho)}{\rho}\frac{\partial}{\partial\phi}\left(A\sin(l\phi) + B\cos(l\phi)\right) \tag{87}$$

$$E_\phi = \mp Z(\hat{\boldsymbol{\rho}}\cdot\mathbf{H}_t) = -\frac{i\mu\omega a^2}{\chi'^2_{nl}}\left(A\sin(l\phi) + B\cos(l\phi)\right)\frac{\partial}{\partial\rho}J_l(\frac{\chi_{nl}}{a}\rho) \ , \tag{88}$$

after using $Z = \mu\omega/k_{nl}$. We can examine the behavior of the fields by investigating the case of azimuthal symmetry ($l = 0$), which has an axial magnetic field given by

$$H_z = H_0 J_0\left(\gamma\rho\right)e^{ikz} \ . \tag{89}$$

This satisfies the boundary condition

$$0 = \left.\frac{\partial}{\partial\rho}J_0(\gamma\rho)\right|_{\rho=a} = -\gamma J_1(\gamma a) \quad \Rightarrow \quad \gamma_{n0} = \frac{\chi_{n1}}{a} \ , \tag{90}$$

[2] An equivalent condition be seen by carrying out the explicit derivative of the Bessel function:

$$0 = \left.\frac{\partial}{\partial\rho}J_l(\gamma\rho)\right|_{\rho=a} = \frac{1}{2}\gamma(J_{l-1}(\gamma a) - J_{l+1}(\gamma a))$$

.

where χ_{n0} is the nth zero of the zero order Bessel function of the first kind. The dispersion relation is given by $k_{n0}^2 = \mu\epsilon\omega^2 - (\chi_{n1}/a)^2$, so the axial magnetic field of this mode is then

$$H_z = H_0 J_0\left(\frac{\chi_{n1}}{a}\rho\right) e^{ik_{n0}z} \ . \tag{91}$$

The transverse magnetic field is

$$\mathbf{H}_t = \pm\frac{ik_{n1}a^2}{\chi_{n1}^2}\frac{\partial}{\partial\rho}J_0(\frac{\chi_{n1}}{a}\rho)\hat{\boldsymbol{\rho}} = \mp\frac{ik_{n0}a}{\chi_{n1}}J_1(\frac{\chi_{n1}}{a}\rho)\hat{\boldsymbol{\rho}} \ , \tag{92}$$

and the transverse electric field is

$$\mathbf{E}_t = \pm\frac{i\mu\omega a^2}{\chi_{n1}^2}\frac{\partial}{\partial\rho}J_0(\frac{\chi_{n1}}{a}\rho)\hat{\boldsymbol{\phi}} = \pm\frac{i\mu\omega a}{\chi_{n1}}J_1(\frac{\chi_{n1}}{a}\rho)\hat{\boldsymbol{\phi}} \ . \tag{93}$$

3.2 Transverse Magnetic (TM) Mode.

The transverse magnetic mode has $H_z = 0$, and the axial electric field is

$$E_z = J_l(\gamma\rho)\left(A\sin(l\phi) + B\cos(l\phi)\right) e^{ikz} \ , \tag{94}$$

with $l \in \mathbb{Z}$. These modes must satisfy the boundary conditions $\psi(\rho = a, \phi) = 0$, which for this functional form implies

$$J_l(\gamma a) = 0 \quad \Rightarrow \quad \gamma a = \chi_{nl} \ , \tag{95}$$

where χ_{nl} is the nth zero of the lth Bessel function of the first kind. Using the definition of γ^2 given earlier (see Jackson equation 8.35) we see the dispersion relation is

$$k_{nl}^2 = \mu\epsilon\omega^2 - \frac{\chi_{nl}^2}{a^2} \ , \tag{96}$$

where now the wave number takes a different value for each mode $\{n, l\}$ for a given frequency ω. We can define a cutoff frequency using Jackson equation 8.38:

$$\omega_{nl} = \frac{\chi_{nl}}{a\sqrt{\mu\epsilon}} \tag{97}$$

The dispersion relation for TM waves can be determined by Jackson equation 8.37:

$$k_{nl}^2 = \mu\epsilon\omega^2 - \frac{\chi_{nl}^2}{a^2} \ , \tag{98}$$

which now replaces k in Equation 94. Using the expression for the transverse components of the electric field (Jackson equation 8.33), we see

$$\mathbf{E}_t = \pm\frac{ik_{nl}}{\chi_{nl}/a}\left(\frac{\partial}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}}\right) J_l(\frac{\chi_{nl}}{a}\rho)\left(A\sin(l\phi) + B\cos(l\phi)\right) \tag{99}$$

(not including the implicit carrier wave $\exp[ik_{n_x n_y}z - i\omega t]$). Using Jackson equation 8.31 the transverse components of the magnetic field can be found:

$$\begin{aligned}\mathbf{H}_t &= \pm\frac{ik_{nl}^2 a}{\epsilon\omega\chi_{nl}}\hat{\mathbf{z}}\times\left(\left(A\sin(l\phi) + B\cos(l\phi)\right)\frac{\partial}{\partial\rho}J_l(\frac{\chi_{nl}}{a}\rho)\hat{\boldsymbol{\rho}} + \frac{J_l(\frac{\chi_{nl}}{a}\rho)}{\rho}\frac{\partial}{\partial\phi}\left(A\sin(l\phi) + B\cos(l\phi)\right)\hat{\boldsymbol{\phi}}\right) \\ &= \pm\frac{ik_{nl}^2 a}{\epsilon\omega\chi_{nl}}\left(\left(A\sin(l\phi) + B\cos(l\phi)\right)\frac{\partial}{\partial\rho}J_l(\frac{\chi_{nl}}{a}\rho)\hat{\boldsymbol{\psi}} - \frac{J_l(\frac{\chi_{nl}}{a}\rho)}{\rho}\frac{\partial}{\partial\phi}\left(A\sin(l\phi) + B\cos(l\phi)\right)\hat{\boldsymbol{\rho}}\right) \ .\end{aligned}$$

In order to solve these to demonstrate the fields' behaviors, we will examine the case of azimuthal symmetry ($l = 0$), which has the z component of the electric field given by

$$E_z = E_0 J_0(\frac{\chi_{n0}}{a}\rho)e^{ik_{n0}z} \ , \tag{100}$$

with $k_{n0}^2 = \mu\epsilon\omega^2 - (\chi_{n0}/a)^2$. The transverse components for these modes are

$$\mathbf{E}_t = \pm E_0 \frac{i\left[\mu\epsilon\omega^2 - (\chi_{n0}/a)^2\right] a}{\chi_{n0}} \frac{\partial}{\partial\rho} J_0(\frac{\chi_{n0}}{a}\rho)\hat{\boldsymbol{\rho}} \ , \tag{101}$$

so

$$E_\rho = \mp E_0 \frac{i\left[\mu\epsilon\omega^2 - (\chi_{n0}/a)^2\right] a}{\chi_{n0}} \frac{\chi_{n0}}{a} J_1(\frac{\chi_{n0}}{a}\rho) = \mp iE_0 \left[\mu\epsilon\omega^2 - (\chi_{n0}/a)^2\right] J_1(\frac{\chi_{n0}}{a}\rho), \tag{102}$$

and

$$H_\phi = \mp \frac{iE_0}{\epsilon\omega} \left[\mu\epsilon\omega^2 - (\chi_{n0}/a)^2\right]^{3/2} J_1(\frac{\chi_{n0}}{a}\rho), \tag{103}$$

not including the implicit carrier wave $\exp[ik_{n_x n_y}z - i\omega t]$.

4 Transverse Electromagnetic (TEM) Modes in Coaxial Waveguide.

Repeat the problem above for a TEM mode in a coaxial waveguide with inner radius a and outer radius b.

The axial components of both the electric and magnetic fields in a TEM mode are zero, so

$$\mathbf{E}_{\mathrm{TEM}} = \mathbf{E}_t + E_z\hat{\mathbf{z}} = \mathbf{E}_t \qquad \mathbf{H}_{\mathrm{TEM}} = \mathbf{H}_t + H_z\hat{\mathbf{z}} = \mathbf{H}_t \ , \tag{104}$$

we can substitute these into the Maxwell equations written in terms of transverse and axial components, given by Jackson equations 8.23-8.25. Given that the axial components are zero: $E_z = H_z = 0$, and using the equation above, these equations reduce to

$$\frac{\partial \mathbf{E}_{\mathrm{TEM}}}{\partial z} + i\mu\omega\hat{\mathbf{z}} \times \mathbf{H}_{\mathrm{TEM}} = 0 \qquad \hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_t \times \mathbf{E}_{\mathrm{TEM}}) = 0 \tag{105}$$

$$\frac{\partial \mathbf{H}_{\mathrm{TEM}}}{\partial z} - i\omega\epsilon\hat{\mathbf{z}} \times \mathbf{E}_{\mathrm{TEM}} = 0 \qquad \hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_t \times \mathbf{H}_{\mathrm{TEM}}) = 0 \tag{106}$$

$$\boldsymbol{\nabla}_t \cdot \mathbf{E}_{\mathrm{TEM}} = 0 \qquad \boldsymbol{\nabla}_t \cdot \mathbf{H}_{\mathrm{TEM}} = 0 \ . \tag{107}$$

Using the right-hand expressions in Equations 105 and 106, we see that

$$\boldsymbol{\nabla}_t \times \mathbf{E}_{\mathrm{TEM}} = 0 \qquad \boldsymbol{\nabla}_t \times \mathbf{H}_{\mathrm{TEM}} = 0 \ , \tag{108}$$

noting that taking the transverse curl (no z component) of a transverse field (no z component) must be entirely in the z direction: $(\boldsymbol{\nabla}_t \times \mathbf{F}_{\mathrm{TEM}}) \parallel \hat{\mathbf{z}}$ (where $\mathbf{F}$ is a generic field, *i.e.*, electric or magnetic) and if the dot product of the resulting vector with the z unit vector is zero, then the transverse curl must be zero. The electric field satisfies Equation 108, so we may define the electric field to be the gradient of a scalar field: $\mathbf{E}_t = -\boldsymbol{\nabla}\psi$. Since the electric field also satisfies Equation 107, it must be that ψ satisfies Laplace's equation

$$\nabla_t^2\psi = 0 \ . \tag{109}$$

Furthermore, we can take the transverse curl of Equation 108 and write

$$\boldsymbol{\nabla}_t \times \boldsymbol{\nabla}_t \times \mathbf{E}_t = \boldsymbol{\nabla}_t(\boldsymbol{\nabla}_t \cdot \mathbf{E}_t) - \nabla_t^2\mathbf{E}_t = \nabla_t^2\mathbf{E}_t = 0 \ , \tag{110}$$

due to Equation 107, so the electric field also satisfies Laplace's equation. Additionally, from the wave equation, we may assume a harmonic form of the field

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0(x, y)e^{ikz - i\omega t} \ , \tag{111}$$

which allows the Helmholtz wave equation (Jackson equation 8.17) to be written

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\mathbf{E}_0 = 0 \ . \tag{112}$$

For this and Equation 109 to be simultaneously satisfied, the constant term in the Helmholtz wave equation must be zero, and we obtain the dispersion relation

$$k = \omega\sqrt{\mu\epsilon} \ , \tag{113}$$

which is the infinite-medium value. We are considering a coaxial geometry, where the only media are perfect conductors or vacuum, so the dispersion relation in the transmission region is simply

$k = \omega\sqrt{\mu_0\epsilon_0} = \omega/c$. We will assume a separable form for the electric field $\psi(\rho, \phi) = R(\rho)\Theta(\phi)$, so Laplace's equation becomes

$$0 = \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]\psi(\rho,\phi) = \Theta(\phi)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) + R(\rho)\frac{1}{\rho^2}\frac{\partial^2\Theta(\phi)}{\partial\phi^2} \ . \tag{114}$$

Multiplying through by ρ^2 and dividing by $R(\rho)\Theta(\phi)$ yields the expression

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) = -\frac{1}{\Theta(\phi)}\frac{\partial^2\Theta(\phi)}{\partial\phi^2} \ , \tag{115}$$

which we can set equal to a constant ℓ^2 and obtain the separated ODEs:

$$\rho\left(\frac{\partial R(\rho)}{\partial\rho} + \rho\frac{\partial^2 R(\rho)}{\partial\rho^2}\right) = \ell^2 R(\rho) \tag{116}$$

$$\frac{\partial^2\Theta(\phi)}{\partial\phi^2} = -\ell^2\Theta(\phi) \ . \tag{117}$$

The second of which has harmonic solutions:

$$\Theta(\phi) = A\cos(\ell\phi) + B\sin(\ell\phi) \ , \tag{118}$$

where if we enforce periodicity: $\Theta(\phi) = \Theta(\phi + 2\pi)$, we constain ℓ to be an integer, as shown in Problem 3. Moving all of the terms from the radial ODE to one side yields the equation

$$\rho^2 R'' + \rho R' - \ell^2 R = 0 \ , \tag{119}$$

where the primes denote derivatives with respect to ρ. This is the Euler equation[3] and has solutions of the form

$$R(\rho) = C\rho^{\ell} + D\rho^{-\ell} \ , \tag{120}$$

except in the special case $l = 0$, which has solutions

$$R_0(\rho) = C_0 + D_0\ln\rho \ . \tag{121}$$

Imagine we hold the inner conductor at a constant potential V_0 and hold the outer conductor at ground. To ensure the potential is continuous, we enforce

$$V_0 = R(a) \qquad V_0 = R_0(a) \tag{122}$$

$$0 = R(b) \qquad 0 = R_0(b) \ , \tag{123}$$

for any ϕ. Let us only consider the lowest mode $l = 0$ (which implies azimuthal symmetry), yielding the boundary conditions

$$V_0 = C_0 + D_0\ln a \tag{124}$$

$$0 = C_0 + D_0\ln b \ . \tag{125}$$

From the second, we see $C_0 = -D_0\ln b$, which makes the first

$$V_0 = D_0(\ln a - \ln b) \quad \Rightarrow \quad D_0 = \frac{V_0}{\ln\left(\frac{a}{b}\right)} \quad \text{so} \quad C_0 = -\frac{V_0\ln b}{\ln\left(\frac{a}{b}\right)} \ . \tag{126}$$

[3] Polyanin, "Second-Order Euler Equation". EqWorld, http://eqworld.ipmnet.ru/en/solutions/ode/ode0212.pdf.

For the $\ell = 0$ TEM mode, the scalar potential is

$$\psi_0(\rho) = \frac{V_0}{\ln\left(\frac{a}{b}\right)}\left(\log\rho - \log b\right) = V_0 \frac{\log(\rho/b)}{\log(a/b)} \ , \tag{127}$$

and thus, the transverse electric field is given by

$$\mathbf{E}_{\mathrm{TEM}} = -\boldsymbol{\nabla}\psi_0(\rho) = -\frac{V_0}{\log(a/b)}\frac{\mathrm{d}}{\mathrm{d}\rho}\log(\rho/b)\hat{\boldsymbol{\rho}} = -\frac{V_0}{\log(a/b)}\frac{1}{\rho}\hat{\boldsymbol{\rho}} \ , \tag{128}$$

we now see that V_0 is proportional to the electric field strength and a geometric factor. Since the wave propagates along the z direction and the electric field only has a radial component, the magnetic field must only have an azimuthal component. More specifically, Jackson equation 8.28 tells us the transverse magnetic field is

$$\mathbf{H}_{\mathrm{TEM}} = \mp\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{V_0}{\log(a/b)}\frac{1}{\rho}\hat{\boldsymbol{\phi}} \ , \tag{129}$$

both fields include the implicit carrier wave.

5 Lumped Element Model.

In class, we have discussed the case of a two-wire transmission line with an inductance and capacitance per unit length. Here we would like to consider another example, that of a resistive wire of length L above a ground plane of negligible resistance. In many circuits, one has devices, modeled here by the wire, whose properties can be modified by applying a voltage to a gate, modeled here by the ground plane the idea being to switch the characteristics as fast a possible (think of switching a transistor on and off). Hence the impedance of the structure is important. The ground plane is separated from the wire by a dielectric, but it is a leaky dielectric, so that there is a finite but small conductance between the ground plane and wire.

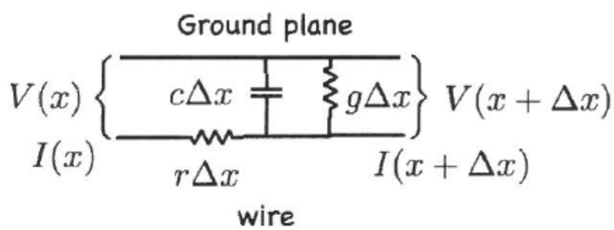

Figure 1: A section of the structure in problem #5.

Using the lumped element model, calculate the admittance Y of structure, if a potential V is applied to the ground plane, and the resulting current I that emanates from one end of the wire is measured (*i.e.*, $Y = I/V$), in terms of C, R, and G. To do this develop a differential equation for either the current of the voltage as a function of position x along the wire, using the capacitance, conductance, and resistance per unit length $c = C/L$, $r = R/L$, and $g = G/L$), as shown in Figure 1.

The current across a resistor of resistance R is $I = V/R$, while the current across a resistor of conductance G is $I = GV$. The current across a capacitor of capacitance C is given by

$$I = \frac{\partial Q}{\partial t} = C\frac{\partial V}{\partial t} \ . \tag{130}$$

To evaluate the circuit shown in Figure 1, it helps to consider the equivalent circuit pictured in Figure 2.

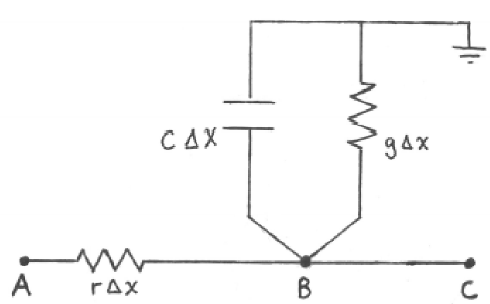

Figure 2: Equivalent circiut to the lumped element model of the transmission line. The voltage at node **A** is $V(x)$ and the current is $I(x)$. At node **B** (the same node as **C**) the voltage is $V(x+\Delta x)$ and the current is $I(x+\Delta x)$.

Let us apply Kirchhoff's current law to the node **B**: the current coming into the node is $I(x)$ and splits into the three branches:

$$I(x) - (c\Delta x)\frac{\partial V(x+\Delta x)}{\partial t} - (g\Delta x)V(x+\Delta x) - I(x+\Delta x) = 0\ . \tag{131}$$

The voltage measured at node **B** is simply the sum of the voltage at node **A** and the voltage drop across the resistor:

$$V(x) - (r\Delta x)I(x) - V(x+\Delta x) = 0\ . \tag{132}$$

To find differential equations, we can use the definition of the derivative[4]. First, divide both equations by Δx:

$$0 = \frac{I(x) - I(x+\Delta x)}{\Delta x} - c\frac{\partial V(x+\Delta x)}{\partial t} - gV(x+\Delta x) \tag{133}$$

$$0 = \frac{V(x) - V(x+\Delta x)}{\Delta x} - rI(x)\ , \tag{134}$$

taking the limit $\Delta x \to 0$, and moving the differentials to the other side yields

$$\frac{\partial I}{\partial x} = -c\frac{\partial V(x)}{\partial t} - gV(x) \tag{135}$$

$$\frac{\partial V}{\partial x} = -rI(x)\ . \tag{136}$$

If we take the time spatial derivative of the second equation,

$$\frac{\partial^2 V}{\partial x^2} = -r\frac{\partial I(x)}{\partial x}\ , \tag{137}$$

and insert the first equation, we obtain the single partial differential equation for the voltage:

$$\frac{\partial^2 V}{\partial x^2} = cr\frac{\partial V(x)}{\partial t} + grV(x)\ . \tag{138}$$

If we assume we apply an AC current of frequency ω to the transmission line, the voltage will oscillate sinusoidally in time ($\sim e^{\pm i\omega t}$). Now carrying out the time derivative yields the ODE

$$\frac{\partial^2 V}{\partial x^2} = (g \pm i\omega c)rV(x) = (G \pm i\omega C)\frac{R}{L^2}V(x)\ , \tag{139}$$

which has solutions of the form

$$V(x) = V_0^{(+)}\exp\left\{\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\} + V_0^{(-)}\exp\left\{-\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\}\ . \tag{140}$$

Using Equation 136, the current at a point x along the line is

$$I(x) = \frac{L}{R}\sqrt{(i\omega C \pm G)R}\frac{1}{L}\left[V_0^{(+)}\exp\left\{\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\} - V_0^{(-)}\exp\left\{-\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\}\right]$$

$$I(x) = \sqrt{\frac{i\omega C \pm G}{R}}\left[V_0^{(+)}\exp\left\{\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\} - V_0^{(-)}\exp\left\{-\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\}\right]$$

[4]The definition of a derivative is $\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$.

We can now find the admittance $Y = I/V$ by taking the ratio of the expression above and Equation 140:

$$Y = \sqrt{\frac{i\omega C \pm G}{R}} \frac{V_0^{(+)} \exp\left\{\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\} - V_0^{(-)} \exp\left\{-\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\}}{V_0^{(+)} \exp\left\{\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\} + V_0^{(-)} \exp\left\{-\left(\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right)\right\}} \tag{141}$$

$$= \sqrt{\frac{i\omega C \pm G}{R}} \tanh\left\{\sqrt{(i\omega C \pm G)R}\frac{x}{L}\right\} \ . \tag{142}$$

We can define the characteristic addmitance $Y_0 = I_0^{(+)}/V_0^{(+)} = -I_0^{(-)}/V_0^{(-)}$, which is given by

$$Y_0 = \sqrt{\frac{i\omega C \pm G}{R}} \ . \tag{143}$$