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Contents

1 Fields of Moving Point Charge.

Starting from the relations between derivatives derived in class:

$$\frac{\partial}{\partial t} = \frac{r}{s}\frac{\partial}{\partial t'} \tag{1}$$

$$\boldsymbol{\nabla}_r = \boldsymbol{\nabla}_1 - \frac{\mathbf{r}}{sc}\frac{\partial}{\partial t'} \ , \tag{2}$$

show that the fields of a moving point charge are given by

$$\frac{4\pi\epsilon_0}{e}\mathbf{E} = \frac{1}{s^3}\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\left(1 - \frac{u^2}{c^2}\right) + \frac{1}{c^2 s^3}\left\{\mathbf{r} \times \left[\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right) \times \dot{\mathbf{u}}\right]\right\} \tag{3}$$

$$\frac{4\pi\epsilon_0 c^2}{e}\mathbf{B} = \frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) + \frac{1}{c^2 s^3}\frac{\mathbf{r}}{r} \times \left\{\mathbf{r} \times \left[\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right) \times \dot{\mathbf{u}}\right]\right\} \ . \tag{4}$$

First let us define a coordinate (and notation) system, let $\mathbf{R}$ be the position of an observation point relative to the origin and $\mathbf{R}'$ the position of the charge. The position of the charge relative to the observation point is then $\mathbf{r} = \mathbf{R} - \mathbf{R}'$ (note un-bolded versions of these symbols correspond to their magnitudes). The present time is t and the retarded time is t', related to the separation by $r = c(t - t')$. The velocity of the particle is $\mathbf{u} = \frac{\mathrm{d}\mathbf{R}'}{\mathrm{d}t'} = -\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t'}$. The scalar and vector potentials due to a moving point charge are

$$\phi(r,t) = \frac{1}{4\pi\epsilon_0}\frac{e}{s} \tag{5}$$

$$\mathbf{A}(r,t) = \frac{\mu_0}{4\pi}\frac{\mathbf{u}e}{s} \ , \tag{6}$$

where s is defined to be

$$s = r - \frac{\mathbf{r}\cdot\mathbf{u}}{c} \ , \tag{7}$$

where $\mathbf{u}$ is the velocity of the point charge. Thus the observed electric field is given by $\mathbf{E} = -\boldsymbol{\nabla}_R\phi - \frac{\partial \mathbf{A}}{\partial t}$ (where $\boldsymbol{\nabla}_R$ denotes derivatives with respect to the observation coordinates). The gradient of the scalar potential is given by

$$-\boldsymbol{\nabla}_R\phi = -\frac{e}{4\pi\epsilon_0}\boldsymbol{\nabla}_R\frac{1}{s} = -\frac{e}{4\pi\epsilon_0}\frac{1}{s^2}\boldsymbol{\nabla}_R s \ , \tag{8}$$

using properties of derivatives. Applying the given derivative relation, and defining $\boldsymbol{\nabla}_1 = \boldsymbol{\nabla}_R|_{t'}$, we obtain

$$-\boldsymbol{\nabla}_R\phi = -\frac{e}{4\pi\epsilon_0}\frac{1}{s^2}\left(\boldsymbol{\nabla}_1 s - \frac{\mathbf{r}}{sc}\frac{\mathrm{d}s}{\mathrm{d}t'}\right) \ , \tag{9}$$

where

$$\boldsymbol{\nabla}_1 s = \boldsymbol{\nabla}_R s|_{t'} = \boldsymbol{\nabla}_R r\bigg|_{t'} - \boldsymbol{\nabla}_R\frac{\mathbf{r}\cdot\mathbf{u}}{c}\bigg|_{t'} \ . \tag{10}$$

Consider the gradient $\boldsymbol{\nabla}_R|\mathbf{R} - \mathbf{R}'|$ of which we inspect the component in the direction of x_i:

$$\left(\boldsymbol{\nabla}_R|\mathbf{R} - \mathbf{R}'|\right)_i = \sum_j \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i\sqrt{(x_j - x_j')^2} = \sum_j \frac{1}{2\sqrt{x_j - x_j'}}2(x_j - x_j')\frac{\partial x_j}{\partial x_i}\hat{\mathbf{e}}_i = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|} = \frac{\mathbf{r}}{r} \ , \tag{11}$$

which gives the frist term in Equation 10. The second term can be found by examining

$$\nabla_R \frac{\mathbf{r}\cdot\mathbf{u}}{c} = \frac{\partial}{\partial x_j}\hat{\mathbf{e}}_j \left(\frac{(R_i - R_i')u_i}{c}\right) = \frac{\partial}{\partial x_j}\hat{\mathbf{e}}_j \left(\frac{R_i u_i}{c} - \frac{R_i' u_i}{c}\right) = \frac{\partial}{\partial x_j}\hat{\mathbf{e}}_j \left(\frac{R_i u_i}{c}\right) \tag{12}$$

$$= \frac{u_i}{c}\hat{\mathbf{e}}_j \delta_{ij} = \frac{\mathbf{u}}{c} \ , \tag{13}$$

and thus the gradient of the scalar potential is

$$-\nabla_R \phi = -\frac{e}{4\pi\epsilon_0}\frac{1}{s^2}\left(\frac{\mathbf{r}}{r}\bigg|_{t'} - \frac{\mathbf{u}}{c}\bigg|_{t'} - \frac{\mathbf{r}}{sc}\frac{\mathrm{d}s}{\mathrm{d}t'}\right) \ . \tag{14}$$

The retarded time derivative of s is given by

$$\frac{\mathrm{d}}{\mathrm{d}t'}\left(r - \frac{\mathbf{r}\cdot\mathbf{u}}{c}\right) = \frac{\mathrm{d}r}{\mathrm{d}t'} - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t'}(\mathbf{r}\cdot\mathbf{u}) = \frac{\mathrm{d}r}{\mathrm{d}t'} - \frac{1}{c}\left\{\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t'}\cdot\mathbf{u} + \mathbf{r}\cdot\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t'}\right\} = \frac{\mathrm{d}r}{\mathrm{d}t'} - \frac{1}{c}\left\{-\mathbf{u}\cdot\mathbf{u} + \mathbf{r}\cdot\dot{\mathbf{u}}\right\} \ , \tag{15}$$

and the retarded time derivative of r is

$$\frac{\mathrm{d}r}{\mathrm{d}t'} = \frac{\mathrm{d}}{\mathrm{d}t'}|\mathbf{R} - \mathbf{R}'| = \frac{\mathrm{d}}{\mathrm{d}t'}\sqrt{(x_i - x_i')^2} = \frac{x_i - x_i'}{\sqrt{(x_i - x_i')^2}}\left(-\frac{\partial x_i'}{\partial t'}\right) = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|}\frac{\partial \mathbf{R}'}{\partial t'} = -\frac{\mathbf{r}}{r}\cdot\mathbf{u} \ . \tag{16}$$

Collecting the results, the scalar potential in Equation 14 can be written

$$-\nabla_R \phi = \frac{e}{4\pi\epsilon_0}\frac{1}{s^2}\left(\frac{\mathbf{r}}{r}\bigg|_{t'} - \frac{\mathbf{u}}{c}\bigg|_{t'} - \frac{\mathbf{r}}{sc}\left[-\frac{\mathbf{r}}{r}\cdot\mathbf{u} - \frac{1}{c}\left\{-\mathbf{u}\cdot\mathbf{u} + \mathbf{r}\cdot\dot{\mathbf{u}}\right\}\right]\right) \tag{17}$$

$$-\frac{4\pi\epsilon_0}{e}\nabla_R \phi = \frac{1}{s^2}\left(\frac{\mathbf{r}}{r} - \frac{\mathbf{u}}{c}\right) - \frac{\mathbf{r}}{s^3 c}\left[-\frac{\mathbf{r}\cdot\mathbf{u}}{r} + \frac{u^2}{c} - \frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\right] \tag{18}$$

The time derivative vector potential is given by

$$-\frac{\partial \mathbf{A}}{\partial t} = -\frac{e}{4\pi\epsilon_0 c^2}\frac{\partial}{\partial t}\left(\frac{\mathbf{u}}{s}\right) = -\frac{e}{4\pi\epsilon_0 c^2}\frac{r}{s}\frac{\partial}{\partial t'}\left(\frac{\mathbf{u}}{s}\right) = -\frac{e}{4\pi\epsilon_0 c^2}\frac{r}{s}\left[\frac{1}{s}\frac{\partial \mathbf{u}}{\partial t'} - \frac{\mathbf{u}}{s^2}\frac{\partial s}{\partial t'}\right] \ . \tag{19}$$

Using the results from Equations 15 and 16, this becomes

$$-\frac{4\pi\epsilon_0}{e}\frac{\partial \mathbf{A}}{\partial t} = -\frac{r}{sc^2}\left[\frac{\dot{\mathbf{u}}}{s} - \frac{\mathbf{u}}{s^2}\left(-\frac{\mathbf{r}\cdot\mathbf{u}}{r} + \frac{u^2}{c} - \frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\right)\right] \tag{20}$$

$$= -\frac{r\dot{\mathbf{u}}}{s^2 c^2} + \frac{r\mathbf{u}}{s^3 c^2}\left(-\frac{\mathbf{r}\cdot\mathbf{u}}{r} + \frac{u^2}{c} - \frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\right) \ , \tag{21}$$

and so the electric field is given by

$$\frac{4\pi\epsilon_0}{e}\mathbf{E} = -\frac{4\pi\epsilon_0}{e}\nabla_R \phi - \frac{4\pi\epsilon_0}{e}\frac{\partial \mathbf{A}}{\partial t} \tag{22}$$

$$= \frac{1}{s^2}\left(\frac{\mathbf{r}}{r} - \frac{\mathbf{u}}{c}\right) + \frac{\mathbf{r}}{s^3 c}\left[\frac{\mathbf{r}\cdot\mathbf{u}}{r} - \frac{u^2}{c} + \frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\right] - \frac{r\dot{\mathbf{u}}}{s^2 c^2} + \frac{r\mathbf{u}}{s^3 c^2}\left(-\frac{\mathbf{r}\cdot\mathbf{u}}{r} + \frac{u^2}{c} - \frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\right) \tag{23}$$

We may multiply both sides by s^3 to obtain

$$s^3\frac{4\pi\epsilon_0}{e}\mathbf{E} = s\left(\frac{\mathbf{r}}{r} - \frac{\mathbf{u}}{c}\right) + \frac{\mathbf{r}}{c}\left[\frac{\mathbf{r}\cdot\mathbf{u}}{r} - \frac{u^2}{c} + \frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\right] - s\frac{r\dot{\mathbf{u}}}{c^2} + \frac{r\mathbf{u}}{c^2}\left(-\frac{\mathbf{r}\cdot\mathbf{u}}{r} + \frac{u^2}{c} - \frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\right) \ , \tag{24}$$

the terms with s remaining are

$$s\left(\frac{\mathbf{r}}{r}-\frac{\mathbf{u}}{c}\right)=\frac{1}{r}\left(\mathbf{r}-\frac{r\mathbf{u}}{c}\right)\left(r-\frac{\mathbf{r}\cdot\mathbf{u}}{c}\right)=\left(\mathbf{r}-\frac{r\mathbf{u}}{c}\right)\left(1-\frac{\hat{\mathbf{r}}\cdot\mathbf{u}}{c}\right)=\mathbf{r}-\frac{\mathbf{r}(\hat{\mathbf{r}}\cdot\mathbf{u})}{c}-\frac{r\mathbf{u}}{c}+\frac{r(\hat{\mathbf{r}}\cdot\mathbf{u})\mathbf{u}}{c^2}$$

and

$$s\frac{r\dot{\mathbf{u}}}{c^2}=\frac{r\dot{\mathbf{u}}}{c^2}\left(r-\frac{\mathbf{r}\cdot\mathbf{u}}{c}\right)=\frac{r^2\dot{\mathbf{u}}}{c^2}-\frac{r(\mathbf{r}\cdot\mathbf{u})\dot{\mathbf{u}}}{c^3}\ . \tag{25}$$

The expression for $(s^3 4\pi\epsilon_0/e)\mathbf{E}$ is now

$$\begin{aligned}\mathbf{r}-\frac{\mathbf{r}(\hat{\mathbf{r}}\cdot\mathbf{u})}{c}-\frac{r\mathbf{u}}{c}+\frac{r(\hat{\mathbf{r}}\cdot\mathbf{u})\mathbf{u}}{c^2}+\frac{\mathbf{r}}{c}\frac{\mathbf{r}\cdot\mathbf{u}}{r}-\frac{\mathbf{r}}{c}\frac{u^2}{c}+\frac{\mathbf{r}}{c}\frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}&\\ -\frac{r^2\dot{\mathbf{u}}}{c^2}+\frac{r(\mathbf{r}\cdot\mathbf{u})\dot{\mathbf{u}}}{c^3}-\frac{r\mathbf{u}}{c^2}\frac{\mathbf{r}\cdot\mathbf{u}}{r}+\frac{r\mathbf{u}}{c^2}\frac{u^2}{c}-\frac{r\mathbf{u}}{c^2}\frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\ ,&\end{aligned} \tag{26}$$

and we will count the terms starting from the left most, indexing from one. Note that terms 2 and 5 cancel, as well as 4 and 10:

$$\mathbf{r}-\frac{r\mathbf{u}}{c}-\frac{\mathbf{r}}{c}\frac{u^2}{c}+\frac{\mathbf{r}}{c}\frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}-\frac{r^2\dot{\mathbf{u}}}{c^2}+\frac{r(\mathbf{r}\cdot\mathbf{u})\dot{\mathbf{u}}}{c^3}+\frac{r\mathbf{u}}{c^2}\frac{u^2}{c}-\frac{r\mathbf{u}}{c^2}\frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\ , \tag{27}$$

collecting terms gives

$$\mathbf{r}\left(1-\frac{u^2}{c^2}\right)-\frac{r\mathbf{u}}{c}\left(1-\frac{u^2}{c^2}\right)+\frac{\mathbf{r}}{c}\frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c}-\frac{r^2\dot{\mathbf{u}}}{c^2}+\frac{r(\mathbf{r}\cdot\mathbf{u})\dot{\mathbf{u}}}{c^3}-\frac{r\mathbf{u}}{c^2}\frac{\mathbf{r}\cdot\dot{\mathbf{u}}}{c} \tag{28}$$

$$\left(\mathbf{r}-\frac{r\mathbf{u}}{c}\right)\left(1-\frac{u^2}{c^2}\right)+\frac{\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{u}})}{c^2}-\frac{(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{u}}}{c^2}+\frac{r(\mathbf{r}\cdot\mathbf{u})\dot{\mathbf{u}}}{c^3}-\frac{r\mathbf{u}(\mathbf{r}\cdot\dot{\mathbf{u}})}{c^3}\ , \tag{29}$$

we can identify the vector triple product rule $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ in the four last terms:

$$\frac{\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{u}})}{c^2}-\frac{(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{u}}}{c^2}=\frac{1}{c^2}\mathbf{r}\times(\mathbf{r}\times\dot{\mathbf{u}}) \tag{30}$$

$$\frac{r(\mathbf{r}\cdot\mathbf{u})\dot{\mathbf{u}}}{c^3}-\frac{r\mathbf{u}(\mathbf{r}\cdot\dot{\mathbf{u}})}{c^3}=\frac{r}{c^3}\mathbf{r}\times(\dot{\mathbf{u}}\times\mathbf{u})=-\frac{r}{c^3}\mathbf{r}\times(\mathbf{u}\times\dot{\mathbf{u}})\ . \tag{31}$$

Combining these results yields

$$s^3\frac{4\pi\epsilon_0}{e}\mathbf{E}=\left(\mathbf{r}-\frac{r\mathbf{u}}{c}\right)\left(1-\frac{u^2}{c^2}\right)+\frac{1}{c^2}\mathbf{r}\times\left[\left(\mathbf{r}-\frac{r\mathbf{u}}{c}\right)\times\dot{\mathbf{u}}\right]\ , \tag{32}$$

which is equivalent to Equation 3.

The magnetic field is given by

$$\mathbf{B}=\boldsymbol{\nabla}_R\times\mathbf{A}=\frac{\mu_0}{4\pi}\boldsymbol{\nabla}_R\times\frac{\mathbf{u}e}{s}=\frac{e}{4\pi\epsilon_0c^2}\boldsymbol{\nabla}_R\times\frac{\mathbf{u}}{s}=\frac{e}{4\pi\epsilon_0c^2}\left(\frac{1}{s}\boldsymbol{\nabla}_R\times\mathbf{u}-\mathbf{u}\times\boldsymbol{\nabla}_R\frac{1}{s}\right)\ . \tag{33}$$

Using the definition of $\boldsymbol{\nabla}_R$ from above, the vector component from above become

$$\frac{1}{s}\left(\boldsymbol{\nabla}_1-\frac{\mathbf{r}}{sc}\frac{\partial}{\partial t'}\right)\times\mathbf{u}+\mathbf{u}\times\left(\frac{1}{s^2}\boldsymbol{\nabla}_1 s-\frac{\mathbf{r}}{s^3c}\frac{\partial s}{\partial t'}\right) \tag{34}$$

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} - \frac{1}{s^2 c}\mathbf{r} \times \dot{\mathbf{u}} + \mathbf{u} \times \left[\frac{1}{s^2 r}\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right) + \frac{\mathbf{r}}{s^3 c}\left(\frac{\mathbf{r}\cdot\mathbf{u}}{r} + \frac{1}{c}\mathbf{r}\cdot\dot{\mathbf{u}} - \frac{1}{c}u^2\right)\right] \ , \tag{35}$$

upon further expansion, we obtain

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} - \frac{1}{s^3 c}\left(r - \frac{\mathbf{r}\cdot\mathbf{u}}{c}\right)\mathbf{r}\times\dot{\mathbf{u}} + \frac{\mathbf{u}\times\mathbf{r}}{s^2 r} - \frac{\mathbf{u}\times\mathbf{u}}{s^2 c} + \left(\frac{\mathbf{r}\cdot\mathbf{u}}{r}\frac{\mathbf{u}\times\mathbf{r}}{s^3 c} + \mathbf{r}\cdot\dot{\mathbf{u}}\frac{\mathbf{u}\times\mathbf{r}}{s^3 c^2} - u^2\frac{\mathbf{u}\times\mathbf{r}}{s^3 c^2}\right)$$

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} - \frac{r(\mathbf{r}\times\dot{\mathbf{u}})}{s^3 c} - \frac{\mathbf{r}\cdot\mathbf{u}}{s^3 c^2}(\mathbf{r}\times\dot{\mathbf{u}}) + \frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{\mathbf{r}\cdot\mathbf{u}}{rc}\right) + \frac{\mathbf{r}\cdot\mathbf{u}}{r}\frac{\mathbf{u}\times\mathbf{r}}{s^3 c} + \mathbf{r}\cdot\dot{\mathbf{u}}\frac{\mathbf{u}\times\mathbf{r}}{s^3 c^2} - u^2\frac{\mathbf{u}\times\mathbf{r}}{s^3 c^2}$$

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} - \frac{r(\mathbf{r}\times\dot{\mathbf{u}})}{s^3 c} + \frac{\mathbf{r}\cdot\mathbf{u}}{s^3 c^2}(\dot{\mathbf{u}}\times\mathbf{r}) + \frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) + \mathbf{r}\cdot\dot{\mathbf{u}}\frac{\mathbf{u}\times\mathbf{r}}{s^3 c^2} \ .$$

Collecting terms yields

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} + \frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) - \frac{1}{s^3 cr}\left[\left(r^2 - r\frac{\mathbf{r}\cdot\mathbf{u}}{c}\right)(\mathbf{r}\times\dot{\mathbf{u}}) - \mathbf{r}\cdot\dot{\mathbf{u}}\frac{r\mathbf{u}\times\mathbf{r}}{c}\right] \tag{36}$$

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} + \frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) - \frac{1}{s^3 cr}\left[(\mathbf{r}\times\dot{\mathbf{u}})\left\{\mathbf{r}\cdot\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\right\} + \mathbf{r}\times\mathbf{u}\frac{r\mathbf{r}\cdot\dot{\mathbf{u}}}{c}\right] \tag{37}$$

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} + \frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) - \frac{1}{s^3 cr}\mathbf{r}\times\left[\dot{\mathbf{u}}\left\{\mathbf{r}\cdot\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\right\} + \left(\frac{\mathbf{u}r}{c}\right)\mathbf{r}\cdot\dot{\mathbf{u}}\right] \ , \tag{38}$$

due to the fact $\mathbf{r}\times\mathbf{r} = 0$ we can insert a term proportional to $\mathbf{r}$ into the square brackets without consequence:

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} + \frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) - \frac{1}{s^3 cr}\mathbf{r}\times\left[\dot{\mathbf{u}}\left\{\mathbf{r}\cdot\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\right\} + \left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\mathbf{r}\cdot\dot{\mathbf{u}}\right] \ , \tag{39}$$

applying the same vector identity as used in the derivation of the electric field, we see this is equivalent to

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} + \frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) - \frac{1}{s^3 c}\frac{\mathbf{r}}{r}\times\left[\mathbf{r}\times\left\{\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\times\dot{\mathbf{u}}\right\}\right] \ . \tag{40}$$

The first term can be expressed as

$$\frac{1}{s}\boldsymbol{\nabla}_1 \times \mathbf{u} = -\frac{1}{s}\boldsymbol{\nabla}_1 \times \frac{\partial \mathbf{r}}{\partial t'} = -\frac{1}{s}\frac{\partial}{\partial t'}\left(\boldsymbol{\nabla}_1 \times \mathbf{r}\right) = 0 \ , \tag{41}$$

noting that $\boldsymbol{\nabla}_1 \times \mathbf{r} = 0$. Using this, we obtain our result

$$\mathbf{B} = \frac{e}{4\pi\epsilon_0 c^2}\left(\frac{\mathbf{u}\times\mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) - \frac{1}{s^3 c}\frac{\mathbf{r}}{r}\times\left[\mathbf{r}\times\left\{\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\times\dot{\mathbf{u}}\right\}\right]\right) \ , \tag{42}$$

or equivalently,

$$\mathbf{B} = \frac{1}{c}\frac{\mathbf{r}}{r}\times\mathbf{E} \ . \tag{43}$$

2 Point Charge with Constant Velocity.

A particle of charge q is traveling at a constant velocity u along the x axis.

2.1 Poynting Vector.

Show that the magnetic field $\mathbf{B}$ due to a moving particle can be written in terms of the electric field $\mathbf{E}$ due to the particle as

$$\mathbf{B} = \frac{1}{c^2}\mathbf{u} \times \mathbf{E} \ , \tag{44}$$

and use this result to calculate the Poynting vector $\mathbf{S}$ entriely in terms of $\mathbf{E}$.

Using Equation 3 for constant velocity $\dot{\mathbf{u}} = 0$, the electric field of a point charge moving at a constant velocity is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{s^3}\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\left(1 - \frac{u^2}{c^2}\right) \ , \tag{45}$$

and similarly by Equation 4, the magnetic field is

$$\mathbf{B} = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{u} \times \mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) \ , \tag{46}$$

note that $\mathbf{u} = u\hat{\mathbf{x}}$. Now consider taking the vector product of $\mathbf{u}$ with the electric field:

$$\frac{1}{c^2}\mathbf{u} \times \mathbf{E} = \frac{q}{4\pi\epsilon_0 c^2}\frac{1}{s^3}\left(\mathbf{u} \times \mathbf{r} - \frac{r\mathbf{u} \times \mathbf{u}}{c}\right)\left(1 - \frac{u^2}{c^2}\right) = \frac{q}{4\pi\epsilon_0 c^2}\frac{1}{s^3}\mathbf{u} \times \mathbf{r}\left(1 - \frac{u^2}{c^2}\right) \ , \tag{47}$$

which is equivalent to Equation 46, thus

$$\mathbf{B} = \frac{1}{c^2}\mathbf{u} \times \mathbf{E} \ . \tag{48}$$

The Poynting vector is then

$$\mathbf{S} = \mathbf{E} \times \frac{\mathbf{B}}{\mu_0} = \frac{1}{\mu_0 c^2}\mathbf{E} \times \mathbf{u} \times \mathbf{E} \ , \tag{49}$$

using the identity for the vector triple product gives

$$\mathbf{S} = \frac{1}{\mu_0 c^2}\left(\mathbf{u}(\mathbf{E} \cdot \mathbf{E}) - \mathbf{E}(\mathbf{E} \times \mathbf{u})\right) = \frac{1}{\mu_0 c^2}\left(\mathbf{u}E^2 - \mathbf{E}(\mathbf{E} \cdot \mathbf{u})\right) \ . \tag{50}$$

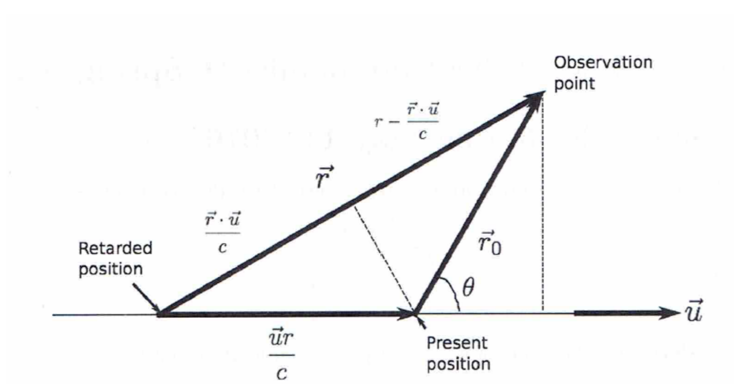

Figure 1: Depiction of geometry for problem #2.

2.2 Direction of Observed Electric Field.

The electric field in part 2.1 involves the factor $s = r - (\mathbf{r}\cdot\mathbf{u})/c$, where $\mathbf{r}$ is the retarded position of the charge with respect to the observation point. Referring to Figure 1, show that s can be expressed in terms of the "present" position" of the charge $\mathbf{r}_0$, *i.e.*, the position of the particle at the instant when the information collection sphere converges on the observation point as

$$s = r_0\sqrt{1 - \frac{u^2}{c^2}\sin^2\theta}\ , \tag{51}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{r}_0$. Show also that the $\mathbf{E}$ is directed along $\mathbf{r}_0$.

Let us define the angle between $\mathbf{r}$ and $\mathbf{r}_0$ as ψ. Applying the law of sines to this triangle, we see

$$\frac{\sin\psi}{r|\mathbf{u}|/c} = \frac{\sin(\pi-\theta)}{r} \quad \Rightarrow \quad \sin\psi = \frac{u}{c}\sin(\pi-\theta) = \frac{u}{c}\sin\theta. \tag{52}$$

Consider the right triangle formed with r_0 as the hypotenuse and $s = r - \mathbf{r}\cdot\mathbf{u}/c$ as one leg, it is evident that

$$s = r_0\cos\psi = r_0\sqrt{1-\sin^2\psi}\ , \tag{53}$$

which if we insert the result from the law of sines, we obtain

$$s = r_0\sqrt{1 - \frac{u^2}{c^2}\sin^2\theta}\ . \tag{54}$$

Using simple vector addition, we see

$$\mathbf{r} - \frac{r}{c}\mathbf{u} = \mathbf{r}_0\ , \tag{55}$$

and then using Equation 45, we see

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{s^3}\left(1 - \frac{u^2}{c^2}\right)\mathbf{r}_0\ , \tag{56}$$

and so $\mathbf{E} \parallel \mathbf{r}_0$.

2.3 Total Power Through a Perpendicular Plane.

Using the result from section 2.2, calculate the total power passing through the plane at $x = a$, at the instant when the particle is at the origin.

The power through the plane is given by

$$\int_A \mathbf{S}\cdot\hat{\mathbf{x}}\ \mathrm{d}A\ , \tag{57}$$

where A is the surface of the plane. Using the result of section 2.1, the Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0 c^2}\left(\mathbf{u}E^2 - \mathbf{E}(\mathbf{E}\cdot\mathbf{u})\right) = \frac{1}{\mu_0 c^2}\left(uE^2\hat{\mathbf{x}} - u\mathbf{E}(\mathbf{E}\cdot\hat{\mathbf{x}})\right)\ . \tag{58}$$

The x component of the Poynting vector is given by

$$S_x = \frac{1}{\mu_0 c^2}\left(uE^2 - u(\mathbf{E}\cdot\hat{\mathbf{x}})^2\right) = \frac{1}{\mu_0 c^2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\left(1-\frac{u^2}{c^2}\right)^2\frac{u}{s^6}\left(r_0^2 - (\mathbf{r}_0\cdot\hat{\mathbf{x}})^2\right)\ . \tag{59}$$

The charge is located at the origin, and we are observing from a plane $x = a$, we then have that $\mathbf{r}_0 \cdot \hat{\mathbf{x}} = r_0 \cos\theta = a$ because $\mathbf{r}_0$ will always terminate at a point on the $x = a$ plane. Let us define a polar coordinate system on the plane such that the origin is located at $x = a, y = z = 0$, with a coordinate ρ as the distance from the origin. Since this geometry is cylindrically symmetric about the particle's velocity (x-axis) we may ignore the polar angle, because when we integrate over the entire plane, we simply acquire a factor of 2π. Simple geometry tells us $r_0^2 = \rho^2 + a^2$, so that the energy flux per unit time through the surface $x = a$ is

$$S_x = \frac{1}{\mu_0 c^2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\left(1 - \frac{u^2}{c^2}\right)^2 \frac{u}{s^6}\left(\rho^2\right) \ , \tag{60}$$

noting that the a^2 from the Pythagorean theorem cancels with the $-a^2$ from the projection of $\mathbf{r}_0$ onto the x axis. Note that there is still dependence on ρ in s - from Equation 54 we have

$$s^2 = r_0^2 - \frac{u^2}{c^2} r_0^2 \sin^2\theta = \rho^2 + a^2 - \frac{u^2}{c^2}\rho^2 \ , \tag{61}$$

noting that $r_0 \sin\theta = \rho$. Finally we have that the x component of the Poynting vector is

$$S_x = \frac{uq^2}{16\pi^2\mu_0\epsilon_0^2 c^2}\left(1 - \frac{u^2}{c^2}\right)^2 \frac{\rho^2}{\left[a^2 + \rho^2\left(1 - \frac{u^2}{c^2}\right)\right]^3} \ , \tag{62}$$

and thus the power through the plane at $x = a$ is

$$P = \int_0^\infty S_x (2\pi\rho \mathrm{d}\rho) = \frac{uq^2}{8\pi\mu_0\epsilon_0^2 c^2}\left(1 - \frac{u^2}{c^2}\right)^2 \int_0^\infty \frac{\rho^3}{\left[a^2 + \rho^2\left(1 - \frac{u^2}{c^2}\right)\right]^3} \mathrm{d}\rho \ , \tag{63}$$

using $c^2 = 1/\mu_0\epsilon_0$, the coefficient simplifies to

$$P = \frac{uq^2}{8\pi\epsilon_0}\left(1 - \frac{u^2}{c^2}\right)^2 \int_0^\infty \frac{\rho^3}{\left[a^2 + \rho^2\left(1 - \frac{u^2}{c^2}\right)\right]^3} \mathrm{d}\rho \ . \tag{64}$$

Letting Mathematica handle the integral, we obtain the result

$$P = \frac{uq^2}{8\pi\epsilon_0}\left(1 - \frac{u^2}{c^2}\right)^2 \frac{1}{4a^2}\left(1 - \frac{u^2}{c^2}\right)^{-2} = \frac{uq^2}{32\pi\epsilon_0 a^2} \ . \tag{65}$$

3 Larmor Formula.

Using the equations derived in class for the radiation field of a moving charged particle, show that the far-field power emitted by a charged electron in the limit $u/c \ll 1$ is given by the Larmor formula

$$-\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{e^2\dot{u}^2}{6\pi\epsilon_0 c^3} \ , \tag{66}$$

where u is the velocity of the electron. Compare this with the time average of the far-field power emitted by an oscillating dipole.

Using the electric field of a moving point charge given by Equation 3, and the result from Equation 43, we see the Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}\mathbf{E}\times\left(\frac{1}{c}\frac{\mathbf{r}}{r}\times\mathbf{E}\right) = \frac{1}{\mu_0 cr}\left[E^2\mathbf{r} - \mathbf{E}(\mathbf{E}\cdot\mathbf{r})\right] \ . \tag{67}$$

Assuming the particle starts from rest, we have $s = r$ and $\mathbf{u} = 0$. Since we are interested in the far-field solution, we can neglect terms of order r^{-2} or more negative powers. With these constraints, the electric field is

$$\frac{4\pi\epsilon_0}{e}\mathbf{E} = \frac{1}{r^3}\mathbf{r} + \frac{1}{c^2r^3}\left\{\mathbf{r}\times\mathbf{r}\times\dot{\mathbf{u}}\right\} \sim \frac{1}{c^2r^3}\left\{\mathbf{r}\times\mathbf{r}\times\dot{\mathbf{u}}\right\} \ , \tag{68}$$

in the far field. We should note that the vector triple product is in the direction of $\mathbf{r}$, so when we take the scalar product indicated in the Poynting vector, we get zero. Therefore

$$\mathbf{S} = \frac{1}{\mu_0 cr}E^2\mathbf{r} = \frac{1}{\mu_0 c}E^2\hat{\mathbf{r}} = \left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{1}{\mu_0 c^5 r^6}\left|\mathbf{r}\times\mathbf{r}\times\dot{\mathbf{u}}\right|^2\hat{\mathbf{r}} \ , \tag{69}$$

if we factor out the magnitude of the separation vector, we obtain

$$\mathbf{S} = \left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{1}{\mu_0 c^5 r^6}\left|r^2\hat{\mathbf{r}}\times\hat{\mathbf{r}}\times\dot{\mathbf{u}}\right|^2\hat{\mathbf{r}} = \left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{\dot{u}^2}{\mu_0 c^5 r^2}\left|\hat{\mathbf{r}}\times\hat{\mathbf{r}}\times\hat{\dot{\mathbf{u}}}\right|^2\hat{\mathbf{r}} \ . \tag{70}$$

If we use the fact that that $\hat{\mathbf{r}} \parallel (\hat{\mathbf{r}}\times\hat{\dot{\mathbf{u}}})$, then

$$\left|\hat{\mathbf{r}}\times\hat{\mathbf{r}}\times\hat{\dot{\mathbf{u}}}\right|^2 = \sin^2\theta, \tag{71}$$

where θ is the angle between the velocity and the separation. As such, the Poynting vector is

$$\mathbf{S} = \left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{\dot{u}^2\sin^2\theta}{\mu_0 c^5 r^2}\hat{\mathbf{r}} \ . \tag{72}$$

The power radiated from the point charge is given by the integral of the power through a spherical shell, of radius r:

$$P = \int \mathbf{S}\cdot\mathrm{d}\mathbf{A} = r^2\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\mathrm{d}\theta\left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{\dot{u}^2\sin^2\theta}{\mu_0 c^5 r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} \ , \tag{73}$$

after integrating over the azimuthal angle, we obtain

$$P = 2\pi\left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{\dot{u}^2}{\mu_0 c^5}\int_0^{\pi}\sin^3\theta\mathrm{d}\theta = 2\pi\left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{\dot{u}^2}{\mu_0 c^5}\frac{4}{3} = \frac{e^2\dot{u}^2}{6\pi\epsilon_0 c^3} \ . \tag{74}$$

Since power is defined as work per unit time, we have

$$-\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{e^2\dot{u}^2}{6\pi\epsilon_0 c^3} \ , \tag{75}$$

which is precisely the field as the average far-field power emitted by an oscillating dipole.

4 Linear Motion of Charged Particle.

Derive the expression for the electric and magnetic fields of a charged particle if the acceleration of the particle is parallel to its velocity. Plot the angular dependence of the radiation flux intensity (the Poynting vector), that is, contours of equal intensity on a polar plot (*i.e.*, for r, θ in polar coordinates) for various values of u/c.

Let us define the z axis to be along the direction of the particle's motion, then $\mathbf{u} = u\hat{\mathbf{z}}$ and $\dot{\mathbf{u}} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \dot{u}\hat{\mathbf{z}}$, and thus

$$\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right) \times \dot{\mathbf{u}} = \mathbf{r} \times \dot{\mathbf{u}} \; . \tag{76}$$

Using this and the results of problem 1, the electric and magnetic fields are given by

$$\frac{4\pi\epsilon_0}{e}\mathbf{E} = \frac{1}{s^3}\left(\mathbf{r} - \frac{r\mathbf{u}}{c}\right)\left(1 - \frac{u^2}{c^2}\right) + \frac{1}{c^2 s^3}\left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \tag{77}$$

$$\frac{4\pi\epsilon_0 c^2}{e}\mathbf{B} = \frac{\mathbf{u} \times \mathbf{r}}{s^3}\left(1 - \frac{u^2}{c^2}\right) + \frac{1}{c^2 s^3}\frac{\mathbf{r}}{r} \times \left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \; . \tag{78}$$

In the radiation zone, we can ignore the first terms because they of order r^{-2}, thus

$$\frac{4\pi\epsilon_0}{e}\mathbf{E} = \frac{1}{c^2 s^3}\left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \tag{79}$$

$$\frac{4\pi\epsilon_0 c^2}{e}\mathbf{B} = \frac{1}{c^2 s^3}\frac{\mathbf{r}}{r} \times \left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \; . \tag{80}$$

Elementary vector analysis on the magnetic field gives

$$\mathbf{r} \times \left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} = \mathbf{r}\left(\mathbf{r} \cdot \{\mathbf{r} \times \dot{\mathbf{u}}\}\right) - \{\mathbf{r} \times \dot{\mathbf{u}}\}\left(\mathbf{r} \cdot \mathbf{r}\right) \tag{81}$$

$$= \mathbf{r}\left(\dot{\mathbf{u}} \cdot \{\mathbf{r} \times \mathbf{r}\}\right) - r^2\{\mathbf{r} \times \dot{\mathbf{u}}\} = r^2\{\dot{\mathbf{u}} \times \mathbf{r}\} \; , \tag{82}$$

and so the electric and magnetic fields of a point charge accelerating in its direction of travel are

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0}\frac{1}{c^2 s^3}\left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \tag{83}$$

$$\mathbf{B} = \frac{e}{4\pi\epsilon_0}\frac{r}{c^2 s^3}\left\{\dot{\mathbf{u}} \times \mathbf{r}\right\} \; . \tag{84}$$

The Poynting vector is then

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{1}{\mu_0}\left(\frac{e}{4\pi\epsilon_0}\right)^2\frac{r}{c^4 s^6}\left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \times \left\{\dot{\mathbf{u}} \times \mathbf{r}\right\} \; , \tag{85}$$

where

$$\left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \times \left\{\dot{\mathbf{u}} \times \mathbf{r}\right\} = -\left\{\dot{\mathbf{u}} \times \mathbf{r}\right\} \times \left\{\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \tag{86}$$

$$= [\mathbf{r} \times \dot{\mathbf{u}}]\left\{\{\dot{\mathbf{u}} \times \mathbf{r}\} \cdot \mathbf{r}\right\} - \mathbf{r}\left\{\{\dot{\mathbf{u}} \times \mathbf{r}\} \cdot [\mathbf{r} \times \dot{\mathbf{u}}]\right\} \tag{87}$$

$$= \mathbf{r}\left|\mathbf{r} \times \dot{\mathbf{u}}\right|^2 = \mathbf{r}(r^2\dot{u}^2\sin^2\theta) = (r^3\dot{u}^2\sin^2\theta)\hat{\mathbf{r}} \; , \tag{88}$$

where θ is the angle between the observation point $\mathbf{r}$ and the z axis. Therefore the Poynting vector is

$$\mathbf{S} = \frac{1}{\epsilon_0\mu_0}\frac{e^2}{16\pi^2\epsilon_0}\frac{r^4}{c^4 s^6}(\dot{u}^2\sin^2\theta)\hat{\mathbf{r}} = \frac{e^2}{16\pi^2\epsilon_0}\frac{r^4}{s^6}\left(\frac{\dot{u}}{c}\right)^2\sin^2\theta\hat{\mathbf{r}} \; , \tag{89}$$

and the flux of this vector through a differential solid angle $\mathrm{d}\Omega$ at radius r, the differential power, is

$$\mathrm{d}P = \mathbf{S}\cdot\mathrm{d}\mathbf{a} = \frac{e^2}{16\pi^2\epsilon_0}\frac{r^4}{s^6}\left(\frac{\dot{u}}{c}\right)^2\sin^2\theta\hat{\mathbf{r}}\cdot r^2\mathrm{d}\Omega\hat{\mathbf{r}} = k\frac{r^6}{s^6}\left(\frac{\dot{u}}{c}\right)^2\sin^2\theta\mathrm{d}\Omega = -\frac{\mathrm{d}W}{\mathrm{d}t} \ . \tag{90}$$

However, the derivative should be evaluated at the retarded time t', so using the fact that $\frac{\mathrm{d}t'}{\mathrm{d}t} = r/s$, we have

$$-\frac{\mathrm{d}W}{\mathrm{d}t'} = -\frac{\mathrm{d}W}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} = k\frac{r^5}{s^5}\left(\frac{\dot{u}}{c}\right)^2\sin^2\theta\mathrm{d}\Omega \ . \tag{91}$$

The distance s is defined as

$$s = r - \frac{\mathbf{r}\cdot\mathbf{u}}{c} = r\left(1 - \frac{\hat{\mathbf{r}}\cdot\mathbf{u}}{c}\right) = r\left(1 - \frac{u}{c}\cos\theta\right) \ , \tag{92}$$

substituting this into the differential power yields expressions for contours of constant intensity:

$$C = \left(\frac{\dot{u}}{c}\right)^2\frac{\sin^2\theta}{\left(1 - \frac{u}{c}\cos\theta\right)^5}\mathrm{d}\Omega \ . \tag{93}$$

5 Energy of Proton from van de Graaff Accelerator.

A proton of charge q is given a constant acceleration in a van de Graaff accelerator by a potential difference of 700 kV. The acceleration region has a length of 3 m. Calculate the ratio of the energy emitted by the proton to its final kinetic energy, and estimate the numerical value of this ratio. Assume the proton starts from rest.

Using the result from problem 3, the total far field energy emitted by an accelerating electron over a time t is given by

$$W = \frac{\mathrm{d}W}{\mathrm{d}t} t = \frac{q^2 \dot{u}^2}{6\pi\epsilon_0 c^3} t \; . \tag{94}$$

Let us assume the acceleration of the proton is constant, which implies the velocity of the particle at time t is simply $u = \dot{u}t$, and as such the length the proton traveled in time t is

$$x = \frac{1}{2}\dot{u}t^2 = \frac{ut}{2} \; , \tag{95}$$

and so the time it takes for a particle to cover a distance x (starting from rest) is $t = 2x/u$. Inserting this into Equation 99, we obtain

$$W = \frac{q^2 x}{3\pi\epsilon_0 c^3}\frac{\dot{u}^2}{u} = \frac{q^2 x}{3\pi\epsilon_0 c^3}\frac{(u/t)^2}{u} = \frac{q^2 x}{3\pi\epsilon_0 c^3}\frac{(u^2/2x)^2}{u} = \frac{q^2}{12\pi\epsilon_0 x}\frac{u^3}{c^3}, \tag{96}$$

which is the energy lost by the proton as it radiates. The proton starts from rest in a potential ϕ, and therefore has potential energy $q\phi$. The energy of the proton when it has velocity u (which it has after it travels for time t/distance x) is $\frac{1}{2}m_p u^2$. If we assume the energy lost to radiation is small, we can conserve energy to find

$$\frac{1}{2}m_p u^2 = q\phi \quad \Rightarrow \quad u = \sqrt{\frac{2q\phi}{m_p}} \; . \tag{97}$$

The ratio of emitted energy to final kinetic energy is

$$\Gamma = \frac{W}{\frac{1}{2}m_p u^2} = \frac{q^2}{6\pi\epsilon_0 x m_p u^2}\frac{u^3}{c^3} = \frac{q^2}{6\pi\epsilon_0}\frac{u}{x m_p c^3} \; , \tag{98}$$

and if we insert our result for the final velocity of the proton, we obtain

$$\Gamma = \frac{q^2}{6\pi\epsilon_0}\frac{1}{x m_p c^3}\sqrt{\frac{2q\phi}{m_p}} = \frac{\sqrt{2}q^{5/2}}{6\pi\epsilon_0 m_p^{3/2} c^3}\frac{\sqrt{\phi}}{x} \; . \tag{99}$$

We may evaluate the physics constants using SI units:

$$\Gamma = (5\times 10^{-23})\left[\frac{\mathrm{s}^{3/2}\cdot \mathrm{A}^{1/2}}{\mathrm{kg}^{1/2}}\right]\frac{\sqrt{\phi}}{x}\left[\frac{\mathrm{V}^{1/2}}{\mathrm{m}}\right] \; , \tag{100}$$

where now ϕ and x are dimensionless. The indicated units cancel exactly, and the ratio Γ is dimensionless, as expected, with a value of

$$\Gamma = (5\times 10^{-23})\frac{\sqrt{7\times 10^5}}{3} = 1.39\times 10^{-20} \; , \tag{101}$$

so our assumption that the energy lost to radiation is small was valid.