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Contents

1 Goldstein 1.13.

1.1 Determining Equation of Motion

To find the equation of motion of a rocket expelling exhaust gasses in a uniform gravitational potential, we begin by determining the momentum of the rocket/fuel system before any fuel (mass) is expelled and after all the fuel has been expelled. The momentum of this system, at time t, is $P_t = P_rocket + P_{fuel}$. Let $t = 0$ be the time after which mass is expelled from the rocket, and $t = \Delta t$ be the time at which all the fuel has been expended. At these times the system's momentum is given by

$$P_0 = (m + \Delta m)v \tag{1}$$

$$P_{\Delta t} = m(v + \Delta v) + v_e \Delta m \ , \tag{2}$$

where m is the mass of the empty rocket, Δm is the mass lost after time Δt (equivalent to the mass of the fuel), and Δv is the velocity gained by the rocket after expelling mass Δm. The velocity of the exhaust gasses in the frame of a stationary observer, v_e, is related to the velocity of the rocket by $v_e = v - u$, where u is the velocity of the gasses relative to the rocket. Substituting this into Equation 2, then subtracting Equation 1 from Equation 2 yields the change in momentum,

$$\Delta P = mv + m\Delta v + v\Delta m - u\Delta m - mv - v\Delta m = m\Delta v - u\Delta m \ . \tag{3}$$

Dividing this by the time interval, Δt, and reducing to infinitesimal quantities gives

$$\frac{dP}{dt} = m\frac{dv}{dt} + u\frac{dm}{dt} \tag{4}$$

where the sign change comes from the fact that expelling a positive Δm results in a decrease in mass, $-dm$. The time derivative of momentum, Equation 4, is also equal to the sum of the external forces, in this case, the force of gravity from the Earth, $\frac{dP}{dt} = -mg$. Using this, and rearranging terms yields the equation of motion,

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - mg \ . \tag{5}$$

1.2 Determining Velocity as a Function of Mass

Dividing Equation 5 by m and integrating with respect to time gives

$$\int_0^t \frac{dv}{dt}dt = -u\int_0^t \frac{dm}{m}\frac{1}{dt}dt - g\int_0^t dt \ . \tag{6}$$

However, since the dt cancels in the first term on the right hand side of Equation 6, the integral is over mass. At time $t = 0$ the mass is $m_0 = m + \Delta m$, and after time t (Δt in Section 1.1) the mass is just m. Let $v(t = 0) = v_0$; integrating the two time integrals and moving v_0 to the right hand side gives an expression for $v(t)$,

$$v(t) = v_0 - u\int_{m_0}^{m} \frac{dm}{m} - gt \ . \tag{7}$$

In order to get this expression as a function of m alone, we can eliminate t, by noting

$$\frac{dm}{dt} = \frac{m - m_0}{t} \ , \tag{8}$$

and solving for t. Evaluating the final integral, combining the logarithms, and substituting in t using the above expression gives the velocity of the rocket as a function of mass,

$$v(m) = v_0 - u \ln \left[\frac{m}{m_0} \right] - g \frac{m - m_0}{\dot{m}} \ . \tag{9}$$

1.3 Determining Fuel to Rocket Mass Ratio

For a rocket initially at rest, with a mass loss per second of $\dot{m} = -m_0/60$, and noting that $m_0 = m + \Delta m$, Equation 9 simplifies to

$$v(m) = -u \ln \left[\frac{m}{m + \Delta m} \right] - g \frac{-\Delta m}{-(m + \Delta m)/60} \ . \tag{10}$$

A rocket reaching escape velocity, V_E, just as it burned off all the fuel (so the final mass is just m), has $v(m) = V_E$. Using this and rearranging Equation 10 gives

$$\ln \left[\frac{m}{m + \Delta m} \right] = -\left(60g \frac{\Delta m}{m + \Delta m} + V_T \right) / u \ , \tag{11}$$

which by exponentiating and inverting simplifies to

$$\frac{m + \Delta m}{m} = \exp \left[\left(60g \frac{\Delta m}{m + \Delta m} + V_T \right) / u \right] \ . \tag{12}$$

The ratio of fuel mass to rocket mass, $\frac{\Delta m}{m}$, can be found using the above equation. Though as it is written, Equation 12 is a transcendental equation, it can be solved if we make the assumption $\Delta m \gg m$, which means $m + \Delta m \simeq \Delta m$. In this case Equation 12 simplifies to

$$\frac{\Delta m}{m} = \exp[(60g + V_T)/u] \ . \tag{13}$$

On Earth, according to the all-knowing Wikipedia, the escape velocity is 11.2 km/s. Assuming the exhaust velocity is 2.1 $km/2$, and $g = 9.8 \ m/s^2$, this ratio is 274.056.

2 Goldstein 1.14.

Both masses, m, are a distance $l/2$ from the center of mass of the two mass and massless rod system. The center of mass of that system is constrained to move along a circle of radius a. Let $\hat{\theta}$ be the angle the center of mass makes with the vertical line through the center of the circle. Let $\hat{\phi}$ be the angle each mass makes with the vertical line through the center of mass (see Figure 1a). This gives position vectors for the center of mass as well as each mass, m_i,

$$\vec{R}_{CM} = a\hat{\theta} \tag{14}$$
$$\vec{r}_i = (l/2)\hat{\phi}\,. \tag{15}$$

Velocities of both masses and center of mass can be found by by differentiating these with respect to time,

$$v_{CM} = a\dot{\theta} \tag{16}$$
$$v_i = (l/2)\dot{\phi}\,. \tag{17}$$

Equation 1.31 of Goldstein tells us the kinetic energy of a system of particles is given by

$$T = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\sum_i m_i v_i^2\,, \tag{18}$$

where M is the total mass of the system, in this case $2m$. Because both masses, their displacement from the center of mass, and their velocities are equivalent, Equation 18 simplifies to

$$T = ma^2\dot{\theta}^2 + mv_i^2\,. \tag{19}$$

Plugging in the velocity of each mass given by Equation 17 yields our final result,

$$T = m[a^2\dot{\theta}^2 + \frac{1}{4}l^2\dot{\phi}^2]\,. \tag{20}$$

3 Problem #3: Path Minimization on a Cylinder.

Cylindrical coordinates, (r, z, θ), on a cylinder of fixed radius, R, reduces to a two dimensional space, $(z, R\theta)$. Assume there is a curve, S with endpoints (z_1, ϕ_1) and (z_2, ϕ_2) (see Figure 1b). Any arc along this curve is given by

$$dS = \sqrt{dz^2 + (Rd\phi)^2}. \tag{21}$$

Using the chain rule,

$$dz = \frac{\partial z}{\partial \phi} d\phi = z' d\phi \,, \tag{22}$$

which gives

$$dS = d\phi \sqrt{[z'(\phi)]^2 + R^2} \,. \tag{23}$$

Integrating this to get the length between the two points: $(z_1, R\phi_1)$ and $(z_2, R\phi_2)$ gives

$$l = \int_{\phi_1}^{\phi_2} d\phi \sqrt{[z'(\phi)]^2 + R^2} \,, \tag{24}$$

where

$$A : z(\phi_1) = z_1 \tag{25}$$
$$B : z(\phi_2) = z_2 \tag{26}$$

are the constraints on the function $z(\phi)$ at points A and B. In these coordinates, the Euler equation is

$$\frac{df}{dz} - \frac{d}{d\phi}\frac{df}{dz'} = 0 \,, \tag{27}$$

where f is given by

$$f(z(\phi), z'(\phi); \phi) = \sqrt{z'(\phi)^2 + R^2} \,. \tag{28}$$

The derivatives of f with respect to z and z' are

$$\frac{df}{dz} = 0 \tag{29}$$
$$\frac{df}{dz'} = \frac{1}{2}(z'(\phi)^2 + R^2)^{-1/2}(2z'(\phi)) \,. \tag{30}$$

Plugging these values into the Euler equation gives

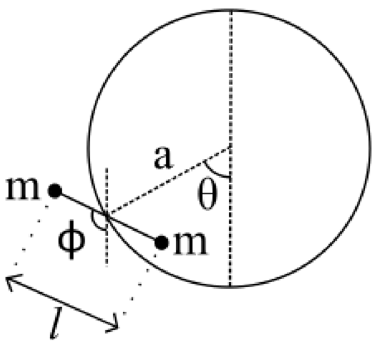
$$\frac{d}{d\phi}\frac{z'(\phi)}{\sqrt{z'(\phi)^2 + R^2}} = 0 \,. \tag{31}$$

This implies the argument for the derivative is equal to a constant, c. Solving this expression for z' in terms of c gives

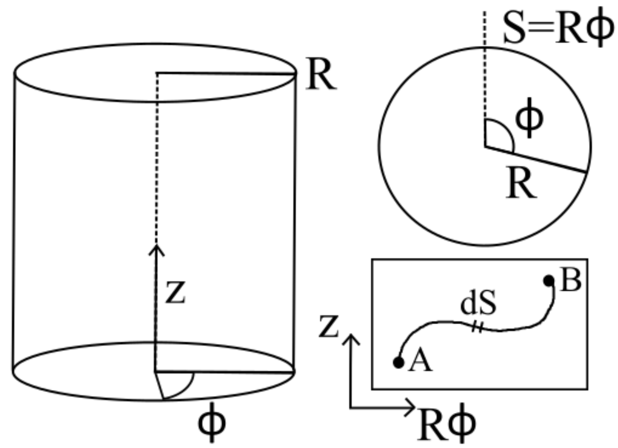
$$z'(\phi) = \frac{cR}{\sqrt{1 - c^2}} \,. \tag{32}$$

Solving this differential equation gives the equation of a helix, $z(\phi) = k\phi$, where k is given by

$$k = \frac{cR}{\sqrt{1 - c^2}} \,. \tag{33}$$



(a) Coordinates for a rod with two masses at each end, constrained to move around a circle - (θ,ϕ).



(b) Coordinates for cylindrical coordinates with a fixed radius - (z,$R\phi$).

Figure 1: Diagrams depicting coordinates for Problems #2 and #3.

4 Problem #4: Motion with Drag.

4.1 Linear Drag

A linear drag coefficient, αv, gives an equation of motion,

$$\frac{dv}{dt} = g - \frac{\alpha}{m} v \, . \tag{34}$$

Through dimensional analysis the units of α can be determined to be $[kg/s]$. In order to find τ in the units of time, the parameters m and α must be combined as

$$\tau = \frac{m}{\alpha} \, . \tag{35}$$

This simplifies the equation of motion to

$$\dot{v} = g - \frac{1}{\tau} v \, . \tag{36}$$

However, when the particle subjected to these forces reaches terminal velocity, V_T, the acceleration is zero. In other words, $\dot{v} = 0$, giving

$$V_T = \tau g \, . \tag{37}$$

This again simplifies the equation of motion to

$$\dot{v} = \frac{V_T}{\tau} - \frac{v}{\tau} \, . \tag{38}$$

In order to make these quantities dimensionless, we make the substitutions

$$u = \frac{v}{V_T} \Rightarrow dv = V_T du \tag{39}$$

$$T = \frac{t}{\tau} \Rightarrow dt = \tau dT \, . \tag{40}$$

Making these substitutions for the derivatives, Equation 38 becomes

$$\frac{du}{dT} \frac{V_T}{\tau} = \frac{1}{\tau} [V_T - v] \, , \tag{41}$$

simplifying to

$$\frac{du}{dT} = 1 - u \, . \tag{42}$$

4.2 Linear Drag

This method is repeated using a quadratic drag factor, βv^2, giving an equation of motion,

$$\frac{dv}{dt} = g - \frac{\beta}{m} v^2 \, . \tag{43}$$

Combining the parameters β, m, and g to find τ in units of time gives

$$\tau = \sqrt{\frac{m}{\beta g}} \, . \tag{44}$$

Meanwhile the terminal velocity, V_T, using the same methodology as in the linear drag case, is given by

$$V_T = \sqrt{\frac{mg}{\beta}} = \tau g \ . \tag{45}$$

Substituting these values into the equation of motion, Equation 43, becomes

$$\frac{dv}{dt} = \frac{V_T}{\tau} - \frac{1}{V_T \tau} v^2 \ , \tag{46}$$

which, after making the substitutions given by Equations 39 and 40

$$\frac{du}{dT} = 1 - u^2 \ . \tag{47}$$

4.3 Numeric Integration

These dimensionless equations of motion can be integrated using computational software. The MATHEMATICA function `NDSolve[]` numerically solves differential equations, and was used to integrate Equations 42 and 47, as follows.

```
linear = NDSolve[{y'[x] == 1 - y[x] , y[0] == 0}, y, {x, 0, 5}];
quad = NDSolve[{y'[x] == 1 - y[x]^2 , y[0] == 0}, y, {x, 0, 5}];
Plot[{Evaluate[y[x] /. linear], Evaluate[y[x] /. quad]}, {x, 0, 5},  PlotRange -> All]
```

This script generates the plot shown in Figure 2, below.

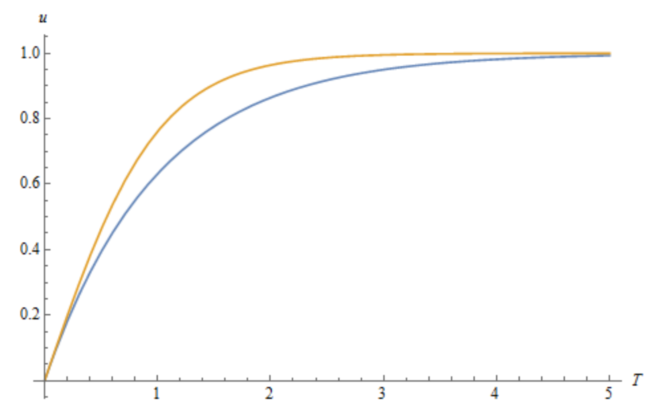


Figure 2: Results of the numerical integration of the dimensionless equations of motion for linear drag (blue) and quadratic drag (yellow).