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Contents

1 Problem #1: Projectile Motion.

Derive the equations of motion for a particle moving in two dimensions under a uniform, vertical gravitational field using the Lagrangian method, in Cartesian and polar coordinates.

1.1 Cartesian Coordinates.

The Lagrangian for two dimensional projectile motion in Cartesian coordinates is simply,

$$\mathcal{L} = T_x + T_y - U_{grav} = \frac{1}{2}mv^2 - mgy = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \ , \tag{1}$$

where m is the mass of the particle and g is the gravitational acceleration. The coordinates are chosen following the convention the x is the horizontal coordinate and y is the vertical. Note that the Lagrangian is cyclic in x, which says $\frac{d\mathcal{L}}{d\dot{x}}$ is a conserved quantity, the linear momentum in the x direction. The Lagrange equations in these coordinates are,

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{x}}\right) = 0 \tag{2}$$

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{y}}\right) - \frac{d\mathcal{L}}{dy} = 0 \ , \tag{3}$$

due to x being a cyclic coordinate. Substituting in the Lagrangian from Equation 1 and performing the derivatives yields,

$$m\ddot{x} = 0 \tag{4}$$

$$m\ddot{y} = \ -mg \ , \tag{5}$$

note the m cancels from each equation. These can be integrated once to get velocities as functions of time,

$$\dot{x} = v_{x0} \tag{6}$$

$$\dot{y} = \ -gt + v_{y0} \ , \tag{7}$$

where v_{i0} is the initial velocity in the i^{th} direction. Upon the second integration, the positions as functions of time are given by,

$$x(t) = x_0 + v_{x0}t \tag{8}$$

$$y(t) = y_0 + v_{y0}t - \frac{1}{2}gt^2 \ , \tag{9}$$

where x_0 and y_0 are the initial horizontal and vertical positions, respectively, which are the classic Newtonian result.

1.2 Polar Coordinates.

In polar coordinates the Lagrangian has the same form as Equation 1 but the components x and y, and their first derivatives have the form,

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad ; \quad \begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases} \quad , \tag{10}$$

which gives the velocity squared to be,

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \ , \tag{11}$$

by noting that the cross terms (ones that contain $\dot{r}$ and $\dot{\theta}$) cancel and using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. Substituting v^2 and y from Equations 11 and 10 into Equation 1 gives the Lagrangian,

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta \ , \tag{12}$$

which is not cyclic in any coordinate, and therefore has no conserved quantities. The Lagrange equations in these coordinates are,

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{r}}\right) - \frac{d\mathcal{L}}{dr} = 0 \tag{13}$$

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) - \frac{d\mathcal{L}}{d\theta} = 0 \ , \tag{14}$$

which gives the equations of motion,

$$m\ddot{r} = mr\ddot{\theta}^2 - mg\sin\theta \tag{15}$$

$$(2mr\dot{r}\dot{\theta}) + (mr^2\ddot{\theta}) = \ - mgr\cos\theta \ , \tag{16}$$

which in a simpler form are,

$$\ddot{r} = r\dot{\theta}^2 - g\sin\theta \tag{17}$$

$$\ddot{\theta} = \ - \frac{g}{r}\dot{\theta}\cos\theta - \frac{2}{r}\dot{r} \ , \tag{18}$$

and can be integrated to get the coordinates r and θ as a function of time.

2 Goldstein 1.21.

Two point masses, m_1 and m_2 are connected by a massless, inextensible string of length ℓ. The mass m_1 rests on the surface of a smooth table, with the string passing through a hole in the center of the table so m_2 hangs suspended. Consider the motion only until m_1 reaches the hole.

2.1 Determining Coordinates.

The mass on the table m_1 is free to move on the two dimensional surface of the table, where the hole is defined as the origin. Additionally, the table surface is defined as the zero level of potential energy. The coordinates that describe the motion of m_1 are standard polar coordinates, r and θ, while m_2 only moves in one direction and can be described by the coordinate y, the distance from the table. The table surface is set to $y = 0$, and y is defined to be positive below the table. Therefore at a positive y position, the potential energy will be negative. Using these coordinates, and the result in Equation 11 for v_1, the velocity of m_1, it is easy to find the kinetic and potential energies,

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{y}^2 \tag{19}$$

$$U = -m_2 g y \ , \tag{20}$$

but by imposing a constraint on the coordinate y due to the length of the string, it can be elimated as follows,

$$y = \ell - r \tag{21}$$

$$\dot{y} = -\dot{r} \ . \tag{22}$$

With this substitution the energies become,

$$T = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2(-\dot{r})^2 = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 \tag{23}$$

$$U = -(m_2 g\ell - m_2 g r) \ , \tag{24}$$

but the constant term in U can be neglected when writing the Lagrangian, because the useful information comes from taking derivatives.

2.2 Finding the Lagrange Equations.

The Lagrangian is,

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 - m_2 g r + m_2 g\ell \ , \tag{25}$$

which is cyclic in θ, corresponding to conservation of angular momentum. This gives Lagrange equations of the form,

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{r}}\right) - \frac{d\mathcal{L}}{dr} = 0 \tag{26}$$

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) = 0 \ , \tag{27}$$

which when substituting in Equation 25 and performing the derivatives gives the equations,

$$\ddot{r} = \left(\frac{m_1}{m_1+m_2}\right) r\dot{\theta}^2 - \left(\frac{m_2}{m_1+m_2}\right) g \tag{28}$$

$$\frac{d}{dt}[m_1 r^2 \dot{\theta}] = 0 \ . \tag{29}$$

By noting that $m_1 r^2 \dot{\theta}$ must equal a constant, call it l for angular momentum, $\dot{\theta}$ can be eliminated from Equation 28,

$$\ddot{r} = \left(\frac{m_1}{m_1+m_2}\right) r \left(\frac{l}{m_1 r^2}\right)^2 - \left(\frac{m_2}{m_1+m_2}\right) g = \left(\frac{l^2}{m_1(m_1+m_2)}\right)\frac{1}{r^3} - \frac{m_2 g}{(m_1+m_2)} \ . \tag{30}$$

which can be solved for $\dot{r}$ by taking the first integral. The equation of motion, Equation 30, can be rewritten as,

$$(m_1+m_2)\ddot{r} = \left(\frac{l^2}{m_1 r^3}\right) - m_2 g \ . \tag{31}$$

2.3 Obtaining the First Integral.

In order to solve Equation 31, an effective potential, V_{eff} is defined to be the the sum of the centrifugal term and true potential term of the Lagrangian. In polar coordinates for this potential, the effective potential is,

$$V_{eff} \equiv m_2 g r - \frac{1}{2} m_1 r^2 \dot{\theta}^2 = m_2 g r - \frac{1}{2} m_1 r^2 \left(\frac{l}{m_1 r^2}\right)^2 = m_2 g r - \frac{1}{2}\frac{l^2}{m_1 r^2} \ , \tag{32}$$

then Equation 31 can be written as,

$$(m_1+m_2)\ddot{r} = -\frac{dV_eff}{dr} \ . \tag{33}$$

A factor of integration $2\dot{r}$ can be applied to Equation 33 making it,

$$2(m_1+m_2)\ddot{r}\dot{r} = -2\left(\frac{dV_eff}{dr}\right)\frac{dr}{dt} \Rightarrow (m_1+m_2)\frac{d}{dt}[\dot{r}^2] = -2\frac{dV_eff}{dt} \ , \tag{34}$$

by noting the time derivative of $\dot{r}^2$ is $2\dot{r}\ddot{r}$. Then integrating each side with respect to time yields,

$$(m_1+m_2)\int \frac{d}{dt}[\dot{r}^2] dt = -2\int \frac{dV_eff}{dt} dt \ , \tag{35}$$

which gives an equation for the radial velocity,

$$\frac{1}{2}(m_1+m_2)\dot{r}^2 = -V_{eff} + C' \ , \tag{36}$$

where C' is a constant of integration, which will be determined by imposing initial conditions on the system. Notice that by moving the V_{eff} term to the other side of the equation yields an expression for total energy, which is equal to a constant and therefore conserved. Substituting in Equation 32 and solving for the radial velocity as a function of r, we obtain the first integral of the equation of motion,

$$\dot{r}(r) = \pm\sqrt{\frac{l^2}{m_1(m_1+m_2)} r^{-2} - \frac{2m_2 g}{(m_1+m_2)} r + C} \ , \tag{37}$$

the constant C' was changed to C because it absorbed the factor of two and the sum of the masses while solving for $\dot{r}$. The two solutions result from taking the square root of $\dot{r}$, and allows selection of the result which has physical meaning. This function tells you how fast m_2 is falling as a function of how close m_1 is to the hole on the table, at a given angular momentum. The constant is necessary to interpret the physical meaning of Equation 37, and to ensure solutions make sense. Consider the initial condition that the system started at rest, with m_1 is located a distance r_0 away from the origin,

$$\begin{cases} \dot{r}(r_0) = 0 \\ l = 0 \ . \end{cases} \tag{38}$$

Under these conditions, Equation 37 can be solved for C, giving

$$C = \frac{2m_2 g}{m_1 + m_2} r_0 \ , \tag{39}$$

which after plugging in, gives the final form for the radial velocity equation,

$$\dot{r}(r) = \pm \sqrt{\frac{l^2}{m_1(m_1+m_2)} r^{-2} + \frac{2m_2 g}{(m_1+m_2)}(r_0 - r)} \ . \tag{40}$$

This equation can be verified by investigating the scenario set as the initial conditions, no angular momentum and no initial velocity. Intuitively, we know the mass on the table should move towards the hole because of gravity pulling down the suspended mass, which implies $\dot{r} < 0$. Additionally, $r < r_0$ for any $t > 0$, because the mass on the table is moving towards the hole. Using these facts, it is easy to see the kernel of the square root is a positive number, which gives a real solution. The negative solution would be selected because of the physical restrictions of the scenario. For any value of r, there is a critical angular momentum l_0 for which the radius of m_1's orbit around the hole will not increase or decrease, $\dot{r} = 0$. This implies m_2 maintains a constant distance from the table. If the angular momentum exceeds l_0, the distance m_1 is from the hole will increase. This critical angular momentum maintains a constant distance of both masses from the hole, given by,

$$\frac{l_0^2}{m_1(m_1+m_2)} r^{-2} = \frac{2m_2 g}{(m_1+m_2)}(r_0 - r) \Rightarrow l_0 = \sqrt{2m_1 m_2 g r^2 (r - r_0)} \ . \tag{41}$$

Clearly this can only happen for r values larger than the initial distance of m_1 to the hole.

3 Goldstein 2.18.

A point mass is constrained to move on a massless hoop of radius a, fixed in a vertical plane and rotating about its vertical symmetry axis with constant angular speed ω. The only external forces on this system arise from gravity.

3.1 Determining Lagrange Equations and Conserved Quantities.

The origin of the coordinate system will be located at the center of the hoop, used in polar coordinates, with a fixed value of r, the radial coordinate. Let θ be the polar angle, the angle the mass makes along the hoop, and ϕ be the azimuthal angle, the angle face of the hoop makes around in its rotation, such that $\omega = \frac{d\phi}{dt}$. In a Cartesian coordinate system, each coordinate and velocity can be defined as,

$$\begin{cases} x = a\sin\theta\cos\phi \\ y = a\sin\theta\sin\phi \\ z = a\cos\phi \end{cases} ; \qquad \begin{cases} \dot{x} = a[(\dot{\theta}\cos\theta)\cos\phi - \sin\theta(\dot{\phi}\sin\phi)] \\ \dot{y} = a[(\dot{\theta}\cos\theta)\sin\phi + \sin\theta(\dot{\phi}\cos\phi)] \\ \dot{z} = -a\dot{\theta}\sin\theta \end{cases} . \tag{42}$$

The Lagrangian is easily written down in a Cartesian coordinate system,

$$\mathcal{L} = \frac{1}{2}mv^2 - mgy = \frac{1}{2}m(x^2 + y^2 + z^2) - mgy \ , \tag{43}$$

but substituting in Equation 42, the Lagrangian in the desired coordinates is obtained,

$$\mathcal{L} = \frac{1}{2}m[a^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)] - mga\cos\theta \ , \tag{44}$$

by noting that in $\dot{x}^2$ and $\dot{\theta}^2$, the cross terms (terms with $\dot{r}\dot{\theta}$) cancel when added. Also note during the calculation of v^2, the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ was used three times. This gives the Lagrange equations,

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) - \frac{d\mathcal{L}}{d\theta} = 0 = \frac{d}{dt}ma^2\dot{\theta} - ma^2\dot{\phi}^2\cos\theta\sin\theta + mga\sin\theta \tag{45}$$

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\phi}}\right) - \frac{d\mathcal{L}}{d\phi} = 0 = \frac{d}{dt}ma^2\dot{\phi}\sin^2\theta \ , \tag{46}$$

and the equations of motion upon substitution of $\omega = \frac{d\phi}{dt}$ and noting it is a constant become,

$$\ddot{\theta} = \omega^2\cos\theta\sin\theta - \frac{g}{a}\sin\theta \tag{47}$$

$$cst = mr^2\omega\sin^2\theta \ . \tag{48}$$

The second equation of motion represents a constant of motion, the quantity $mr^2\omega\sin^2\theta$ is conserved. The Hamiltonian is also conserved, because the Lagrangian does not explicitly depend on time.

3.2 Finding Equilibrium Points.

Again, an effective potential V_{eff} is defined to be the sum of the true potential and centrifugal terms of the Lagrangian. In spherical coordinates, for this potential, the effective potential is,

$$V_{eff} \equiv mga\cos\theta - \frac{1}{2}ma^2\omega^2\sin^2\theta \ . \tag{49}$$

Minimizing this potential with respect to θ will give the values of θ for which equilibrium points exits. Physically, these equilibrium points are locations along the hoop the mass can sit at which, for a given ω, will remain without sliding around the hoop in the $\hat{\boldsymbol{\theta}}$ direction. To minimize V_{eff}, it is differentiated with respect to θ and solved for θ when the derivative equals zero,

$$-\frac{dV_{eff}}{d\theta} = 0 = ma^2\omega^2\sin\theta\cos\theta + mga\sin\theta \tag{50}$$

$$0 = a\omega^2\cos\theta + g \tag{51}$$

$$\cos\theta = -\frac{g}{a\omega^2} \ , \tag{52}$$

which can only be satisfied for $\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$. Define a critical angular speed $\omega_0 = \frac{g}{a}$, so that,

$$\cos\theta = -\frac{\omega_0^2}{\omega^2} \ . \tag{53}$$

For angular velocities greater than ω_0, the above equation is solvable and there exist θ's for stable orbits that are not located at the bottom of the hoop, $\theta = \pi$. When $\omega = \omega_0$ the ratio is -1, which corresponds to the only stable location for the mass to remain is the bottom of the hoop. For $\omega < \omega_0$ the ratio in Equation 53 is greater than one and the equation is not solvable. When this is the case, there is no global minimum of the V_{eff} function in the domain of possible θ values, $\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$, the bottom half of the hoop. The minimum, then, is the location at which $\cos\theta$ takes its minimum value. However, in solving Equation 50, there is another solution, $\sin\theta = 0$. Which happens (in the domain of the possible θ values) at $\theta = \pi$, which is also when $\cos\theta$ takes its minimum value. This implies for $\omega < \omega_0$, the only stable point along the hoop, is at $\theta = \pi$, the bottom of the hoop.

4 Goldstein 3.11.

Two particles move about each other in circular orbits under the influence of gravitational forces with a period τ. Their motion is suddenly stopped at a given instant in time, and they are then released and allowed to fall into each other. Prove that they collide after a time $\tau/(4\sqrt{2})$. The Lagrangian for a system of two particles, in the center of mass frame, where the only forces present are the interactions of the particles with each other reduces to an equivalent one-body problem. This formula is given by Goldstein Equation 3.6,

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{54}$$

$$= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \left(\frac{-Gm_1m_2}{r}\right) , \tag{55}$$

where r is the distance between the masses, and $\mu = \frac{m_1m_2}{m_1+m_2}$, the reduced mass. This Lagrangian is cyclic in θ which implies angular momentum is conserved, as shown in the Lagrange equations,

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{r}}\right) - \frac{d\mathcal{L}}{dr} = 0 = \mu\ddot{r} - [\mu r\dot{\theta}^2 - Gm_1m_2r^{-2}] \tag{56}$$

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) = 0 = \frac{d}{dt}\mu r^2\dot{\theta} . \tag{57}$$

The conserved angular momentum is $l = \mu r^2\dot{\theta}$. The angular velocity of a circular orbit with a given period is $\dot{\theta} = \frac{2\pi}{\tau}$, which when substituted into Equation 56 and simplified, yields,

$$\ddot{r} = \left(\frac{2\pi}{\tau}\right)^2 r - G(m_1 + m_2)r^{-2} . \tag{58}$$

While the two masses are orbiting their common center of mass, the orbit is stable so the distance between the masses doesn't change: $\ddot{r} = \dot{r} = 0$, on this orbit the separation of the masses is r_0. Solving Equation 58 under these conditions gives,

$$r_0^3 = \frac{G(m_1 + m_2)\tau^2}{4\pi^2}. \tag{59}$$

Once the orbits are stopped, $\dot{\theta} = 0$, so Equation 56 reduces to,

$$\ddot{r} = -G(m_1 + m_2)r^{-2} , \tag{60}$$

by dividing by μ. Multiplying this equation by an integrating factor, $2\dot{r}$, becomes,

$$2r\ddot{r} = 2G(m_1 + m_2)(-\dot{r}r^{-2}) \Rightarrow \frac{d}{dt}[\dot{r}^2] = 2G(m_1 + m_2)\frac{d}{dt}[r^{-1}] , \tag{61}$$

taking the integral with respect to time gives an equation for the radial speed as a function of r,

$$\dot{r}^2 = 2G(m_1 + m_2)r^{-1} + C , \tag{62}$$

where C is a constant of integration. This constant can be calculated by noting that at $r = r_0$ the radial velocity, $\dot{r}$ is zero,

$$\dot{r}^2(r_0) = 0 = \frac{2G(m_1 + m_2)}{r_0} + C , \tag{63}$$

and solving for C. This makes the radial velocity, Equation 62 into,

$$\dot{r}(r)^2 = 2G(m_1+m_2)r^{-1} - \frac{2G(m_1+m_2)}{r_0} = 2G(m_1+m_2)\left[\frac{1}{r} - \frac{1}{r_0}\right] = 2G(m_1+m_2)\left[\frac{r_0 - r}{rr_0}\right] , \quad (64)$$

from this we can get an expression for time by noting $\dot{r} = \frac{dr}{dt}$, then inverting and integrating both sides,

$$t = -\int_{r_0}^{0} \frac{dt}{dr} dr = -\sqrt{\frac{1}{2G(m_1+m_2)}} \int_{r_0}^{0} \sqrt{\frac{rr_0}{r_0 - r}} dr \ . \quad (65)$$

The negative sign comes from taking the square root of $\dot{r}^2$, since the masses are attracted to each other r is decreasing as a function of time. Evaluating this integral in Mathematica yields a value for the time of collapse,

$$t = -\sqrt{\frac{1}{2G(m_1+m_2)}} \frac{-\pi r_0^{3/2}}{2} \ . \quad (66)$$

Plugging in the expression for r_0, Equation 59, gives the expression for the time from when the circular orbit stops and the masses collide,

$$t = \frac{\pi}{2}\sqrt{\frac{G(m_1+m_2)\tau^2}{4\pi^2}} \frac{1}{\sqrt{2G(m_1+m_2)}} = \frac{\tau}{4\sqrt{2}} \ , \quad (67)$$

which is the value it should be, as asserted in the beginning.

5 Problem #5: Particle Motion in Potentials.

One can think of motion on the curve surface as motion in an effective potential $U(r)$, where different physical situations correspond to different potentials. Consider the potential energy function

$$U(r) = U_0 - \frac{1}{2}\kappa R^2 \exp[-r^2/R^2] \ , \tag{68}$$

where U_0, κ, and R are all constants. The force a particle feels under this potential is the negative gradient of the potential. For this potential, the force is

$$\mathbf{F} = -\nabla U = \frac{\partial}{\partial r}U\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}U\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}U\hat{\boldsymbol{\phi}} = -\kappa r \exp[-r^2/R^2]\hat{\mathbf{r}} \ , \tag{69}$$

which does not depend on the constant U_0. Physically, this is because one is free to arbitrarily set the level of zero potential. Raising or lowering the potential by a constant value has no effects on the dynamics of a particle subject to that potential. Mathematically, this is true because constants disappear under differentiation. See Figures 1, 2 and 3 for a vizualization of this potential and the effects of varying κ and R on the potential.

A conservative force does the same amount of work moving a particle from a to b regardless of choice of path. Which implies the work done along any closed path is zero. Consider a circle with point a and point b separated by an angle $\theta = \pi$ in a vector force field $\mathbf{F}$. The work done in moving a particle from a to b along the contour of the circle is

$$W_{a\to b} = \int_a^b \mathbf{F}\cdot d\mathbf{r} \ , \tag{70}$$

while the work done moving the particle back from b to a along the other half of the circle is

$$W_{b\to a} = \int_b^a \mathbf{F}\cdot d\mathbf{r} = -W_{a\to b} \ . \tag{71}$$

The particle has now moved along a contour C which bounds the surface of the circle S. The work in moving a particle around C from a through b and back to a is just the sum of the work done in moving the particle across each path,

$$W_C = \oint_C \mathbf{F}\cdot d\mathbf{r} = W_{a\to b} + W_{b\to a} \ , \tag{72}$$

which is clearly zero, which was stated above. Using Stokes' Theorem on Equation 72,

$$0 = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{F})dS \ , \tag{73}$$

which states the curl of a conservative force's vector field is zero (irrotational vector field). The curl of a gradient is always zero if the first and second derivatives are continuous [1], so by finding the force by taking the gradient of a potential guarantees a conservative force. More concisely, if a scalar potential energy function that describes the force exists, the force is conservative by definition. Therefore, the formulation of the question ensures that the force given by Equation 69 is conservative.

[1] Nykamp DQ , "The curl of a gradient is zero." From *Math Insight*. http://mathinsight.org/curl_gradient_zero

The potential energy of a particle in this potential in a small region around the origin, ρ, such that $\rho \ll R$ can be found by Taylor Expanding the potential around the origin. Equation 68 becomes, to second order,

$$U(r) = U(0) + U'(0)r + \frac{1}{2}U''(0)r^2 + O(r^3) \ , \tag{74}$$

where the derivatives are given by,

$$\begin{cases} U'(r) = \kappa r \exp[-r^2/R^2] \\ U''(r) = \kappa \exp[-r^2/R^2] - 2\kappa \frac{r^2}{R^2} \exp[-r^2/R^2] \end{cases} \quad ; \quad \begin{cases} U'(0) = 0 \\ U''(0) = \kappa \end{cases} \ . \tag{75}$$

This makes Equation 76 into,

$$U(r) = U_0 - \frac{1}{2}\kappa R^2 + \frac{1}{2}\kappa r^2 + O(r^3) \ , \tag{76}$$

which has the form of a harmonic oscillator, $V_{HO} = \frac{1}{2}k\rho^2 + V_0$. This form is valid for the small neighborhood around the origin, $-\rho \leq x \leq \rho$. If a particle of mass m is moving in this potential, it has the Lagrangian,

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\kappa x^2 - V_0 \ , \tag{77}$$

which gives the equation of motion,

$$\ddot{x} = -\frac{\kappa}{m}x \ , \tag{78}$$

which can be solved for velocity, v, by imposing initial conditions on the particle's motion. For convenience, define $\omega = \sqrt{\frac{\kappa}{m}}$. At time $t = 0$, the particle passes through the origin with a speed v_0. The velocity as a function of displacement can be found by performing the first integration. Some manipulation of Equation 78 yields

$$\frac{d\dot{x}}{dt} = -\omega^2 x = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{d\dot{x}}{dx}\dot{x} \ , \tag{79}$$

with some rearranging and taking an integral on both sides, this becomes

$$\int_{v_0}^{v(\rho)} \dot{x}d\dot{x} = -\omega^2 \int_0^{\rho} xdx \Rightarrow \frac{1}{2}(v(x)^2 - v_0^2) = -\frac{1}{2}\omega^2\rho^2 \ , \tag{80}$$

which yields the equation for velocity as a function of displacement ρ,

$$v(\rho) = \pm\sqrt{v_0^2 - \omega^2\rho^2} \ . \tag{81}$$

The two solutions make sense because for any given displacement the particle can be travelling in either direction.

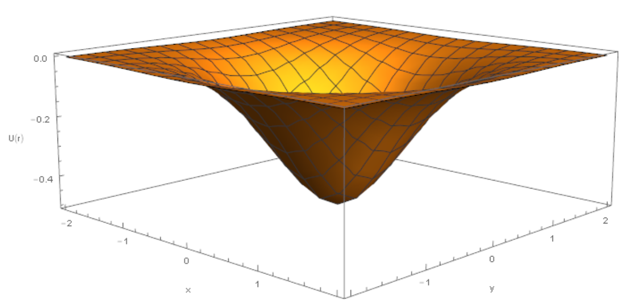
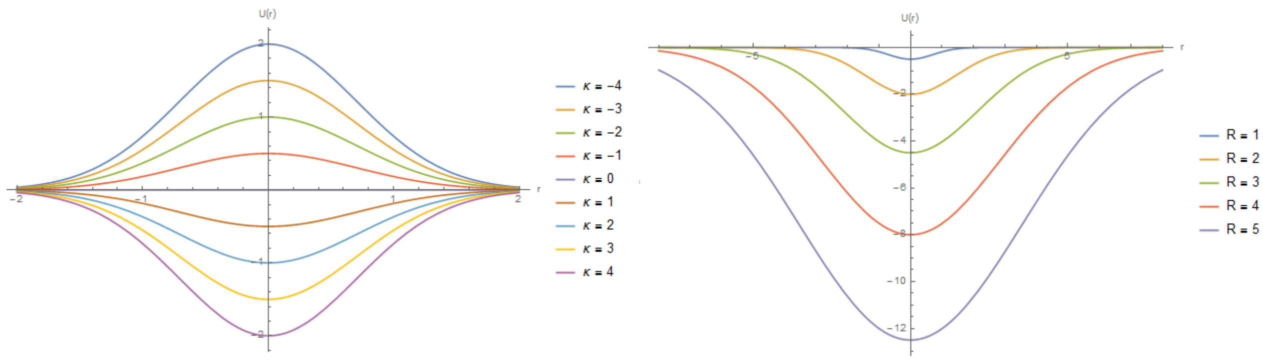


Figure 1: Shape of the potential in two dimensional space. Variables were changed from r to x and y such that $r^2 = x^2 + y^2$. The U_0 parameter was set to zero, because it does not change the shape of the potential, just provides an offset. Both parameters κ and R were set to 1 for this plot.



(a) Plot showing the effects of varying κ from -4 to 4 with integer steps. The other parameters were set to $U_0 = 0$ and $R = 1$. Positive and negative values of κ were chosen because it will change the potential from being repulsive to attractive if $\kappa < 0$ is chosen.

(b) Plot showing the effects of varying R from 1 to 5 with integer steps. The other parameters were set to $U_0 = 0$ and $\kappa = 1$. Only positive values of R were chosen because it only appears in the potential as R^2, so $\pm R$ has the same effect.

Figure 2: Diagrams showing the effects of varying κ and R, respectively.

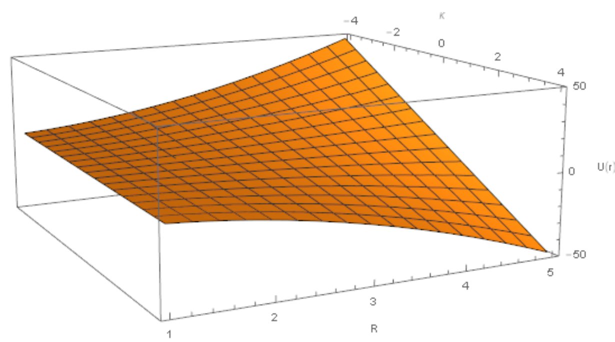


Figure 3: Shape of the potential in parameter space, for $r = 1$ and $U_0 = 0$. Parameter ranges are $-4 \leq \kappa \leq 4$ and $1 \leq R \leq 5$.

The MATHEMATICA code to generate these plots is as follows:

```
U[r_, \[Kappa]_, R_] := - (1/2) \[Kappa] R^2 Exp[-r^2/R^2];
Plot3D[U[Sqrt[x^2 + y^2], 1, 1], {x, -2, 2}, {y, -2, 2}, AxesLabel -> {"x", "y", "U(r)"}]
\[Kappa]List = Table[U[r, \[Kappa], 1], {\[Kappa], -4, 4}];
\[Kappa]Labels = Table["\[Kappa] = " <> ToString[\[Kappa]], {\[Kappa], -4, 4}];
Plot[\[Kappa]List, {r, -2, 2}, PlotLegends -> \[Kappa]Labels, AxesLabel -> {"r", "U(r)"}]
RList = Table[U[r, 1, R], {R, 1, 5}];
RLabels = Table["R = " <> ToString[R], {R, 1, 5}];
Plot[RList, {r, -8, 8}, PlotLegends -> RLabels, AxesLabel -> {"r", "U(r)"}]
Plot3D[U[2, k, R], {k, -4, 4}, {R, 1, 5}, AxesLabel -> {"\[Kappa]", "R", "U(r)"}]
```