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## Contents

# 1 Problem #1: Oblique Dumbbell.

A dumbbell is formed by connecting two small spherical masses of mass $m$ with a massless rod of length $2b$. The rod is attached to an axle in such a way that it makes a constant angle $\phi$ with the axle. The dumbbell rotates about the axle at a rate $\omega$, as shown in Figure 1. Using this information, the moment of inertia tensor $\hat{I}$ and the angular momentum vector $\boldsymbol{L}$ can be found. The coordinate system will be defined such that $\hat{\mathbf{z}}$ points up in Figure 1, the other two coordinates $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ create a right handed Cartesian coordinate system with the origin at the point on the barbell that intersects the vertical line in Figure 1, half the length of the dumbbell. Let the angle the masses are around the axis of rotation be denoted by $\theta = \omega t$. Let $\theta = 0$ point along $\hat{\mathbf{x}}$, thus

$$r_x = b \sin\phi \cos\omega t \tag{1}$$
$$r_y = b \sin\phi \sin\omega t \tag{2}$$
$$r_z = b \cos\phi \tag{3}$$

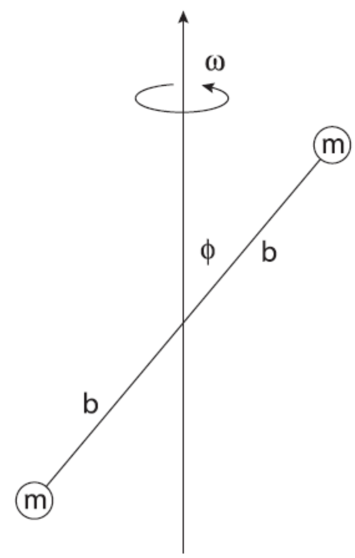


Figure 1: Depiction of the oblique dumbbell system examined in problem #1.

## 1.1 Moment of Inertia Tensor.

The definition of the components of the moment of inertia tensor of a system of discrete masses components is

$$I_{ij} = \sum_{\alpha} m_\alpha \left[ r_{(\alpha)}^2 \delta_{ij} - r_{(\alpha)i} r_{(\alpha)j} \right] \ , \tag{4}$$

where $\delta_{ij}$ is the Kronecker delta and $r^2$ is the Cartesian distance. In this system, the masses are always a constant distance from the origin, and make the same angle with the vertical axis. For this two mass system the moment of inertia components are

$$I_{ij} = m \left[ b^2 \delta_{ij} - r_{(1)i} r_{(1)j} \right] + m \left[ b^2 \delta_{ij} - r_{(2)i} r_{(2)j} \right] \ , \tag{5}$$

but $\mathbf{r}_{(1)} = -\mathbf{r}_{(2)}$, and $r_i = b\hat{\mathbf{e}}_i$, so

$$I_{ij} = 2mb^2 \left[ \delta_{ij} - \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \right] \ . \tag{6}$$

Therefore the diagonal components are

$$I_{11} = 2mb^2[1 - \sin^2\phi \cos^2\omega t] \tag{7}$$
$$I_{22} = 2mb^2[1 - \sin^2\phi \sin^2\omega t] \tag{8}$$
$$I_{33} = 2mb^2[1 - \cos^2\phi] = 2mb^2 \sin^2\phi \ , \tag{9}$$

and the off-diagonal are

$$I_{12} = I_{21} = -2mb^2[\sin^2\phi\cos\omega t\sin\omega t] \tag{10}$$

$$I_{13} = I_{31} = -2mb^2[\sin\phi\cos\phi\cos\omega t] = -2mb^2\cos\omega t\left(\frac{1}{2}\sin 2\phi\right) \tag{11}$$

$$I_{23} = I_{32} = -2mb^2[\sin\phi\cos\phi\sin\omega t] = -2mb^2\sin\omega t\left(\frac{1}{2}\sin 2\phi\right)\ , \tag{12}$$

which makes the moment of inertia tensor for the oblique dumbbell

$$\hat{I} = mb^2\begin{bmatrix} 2-2\sin^2\phi\cos^2\omega t & -2\sin^2\phi\cos\omega t\sin\omega t & -\sin 2\phi\cos\omega t \\ -2\sin^2\phi\cos\omega t\sin\omega t & 2-2\sin^2\phi\sin^2\omega t & -\sin 2\phi\sin\omega t \\ -\sin 2\phi\cos\omega t & -\sin 2\phi\sin\omega t & 2\sin^2\phi \end{bmatrix}\ . \tag{13}$$

## 1.2 Angular Momentum Vector.

From the definition of the coordinate system, the angular velocity vector is

$$\boldsymbol{\omega} = \{0, 0, \omega\}\ , \tag{14}$$

and the definition of angular momentum is

$$\mathbf{L} = \hat{I}\cdot\boldsymbol{\omega}\ . \tag{15}$$

For this system, this is

$$\mathbf{L} = mb^2\begin{bmatrix} 2-2\sin^2\phi\cos^2\omega t & -2\sin^2\phi\cos\omega t\sin\omega t & -\sin 2\phi\cos\omega t \\ -2\sin^2\phi\cos\omega t\sin\omega t & 2-2\sin^2\phi\sin^2\omega t & -\sin 2\phi\sin\omega t \\ -\sin 2\phi\cos\omega t & -\sin 2\phi\sin\omega t & 2\sin^2\phi \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \tag{16}$$

$$\mathbf{L} = m\omega b^2\begin{bmatrix} -\sin 2\phi\cos\omega t \\ -\sin 2\phi\sin\omega t \\ 2\sin^2\phi \end{bmatrix}\ , \tag{17}$$

which is not parallel to $\boldsymbol{\omega}$.

# 2 Problem #2: Binary Star System.

Consider a binary star system, where the two stars are idealized to be point masses of equal mass $m$. Suppose the stars are on a circular orbit, where the separation between them is $r_0$. Due to the fact both stars are equal mass, they both orbit their center of mass on circles of equal radius, $r_0/2$, and with the same period/frequency. They are therefore always at exact opposite locations in their orbit.

## 2.1 Orbital Frequency.

Kepler's Third Law can be written, in the general case of two bodies with different masses, as

$$\frac{\tau^2}{4\pi^2} = \frac{a^3}{G(m_1+m_2)} \ , \tag{18}$$

where $a$ is the semi-major axis of the orbit. For the binary system in question, this becomes

$$\frac{1}{\omega^2} = \frac{(r_0/2)^3}{2Gm} \ , \tag{19}$$

which gives the orbital frequency to be $\omega = \sqrt{2^4 Gm/r_0^3}$.

## 2.2 Planar Locations.

The two stars orbit their center of mass, which means the three dimensional system can be reduced to the motion in a plane. Consider a coordinate frame such that at $t = 0$ one star lies at $(r_0/2)\hat{\mathbf{x}}$ and the other at $-(r_0/2)\hat{\mathbf{x}}$. Let $\hat{\mathbf{y}}$ be perpendicular to this, making a right handed frame. As stated previously, the stars are always at exact opposite points along the orbit so $\{x_1, y_1\} = \{-x_2, -y_2\}$. Using this fact and the initial condition, the coordinates for each star as a function of time are

$$\{x, y\} \equiv \{x_1, y_1\} = \{(r_0/2)\cos\omega t, (r_0/2)\sin\omega t\} \tag{20}$$

$$\{x_2, y_2\} = \{-(r_0/2)\cos\omega t, -(r_0/2)\sin\omega t\} = \{-x_1, -y_1\} \ . \tag{21}$$

## 2.3 Moment of Inertia Tensor.

Using the definition of the components of the moment of inertia tensor for a discrete system of point masses, given by Equation 4, the components in this system are

$$I_{ij} = 2m\left[\left(\frac{r_0}{2}\right)^2 \delta_{ij} - r_i r_j\right] \ : \tag{22}$$

$$I_{11} = 2m\left[\left(\frac{r_0}{2}\right)^2 - x^2\right] = 2m\left(\frac{r_0}{2}\right)^2 [1-\cos^2\omega t] = 2m\left(\frac{r_0}{2}\right)^2 \sin^2\omega t \tag{23}$$

$$I_{22} = 2m\left(\frac{r_0}{2}\right)^2 \cos^2\omega t \tag{24}$$

$$I_{12} = -2mxy = -2m\left(\frac{r_0}{2}\right)^2 \cos\omega t\sin\omega t = -2m\left(\frac{r_0}{2}\right)^2\left[\frac{1}{2}\sin 2\omega t\right] \tag{25}$$

$$I_{21} = -m\left(\frac{r_0}{2}\right)^2 \sin 2\omega t \tag{26}$$

which makes the moment of inertia tensor for this binary system

$$\hat{I} = m\left(\frac{r_0}{2}\right)^2 \begin{bmatrix} 2\sin^2\omega t & -\sin 2\omega t & 0 \\ -\sin 2\omega t & 2\cos^2\omega t & 0 \\ 0 & 0 & 2 \end{bmatrix} , \tag{27}$$

because a third dimension is required due to the axis of rotation. However, for rotations about this axis, there is no motion in the $\hat{z}$ direction, so $z = 0$ in all integrals.

# 3 Problem #3: Party Tricks with Cubes

Consider a six sided die , balanced on edge and and then falling over. Assume the die is of uniform density, total mass $m$, and edges of length $a$. Let the edge of the die sit on the $z-y$ plane, with the edge parallel to the $\hat{\mathbf{z}}$ direction. The $\hat{\mathbf{x}}$ direction points perpendicular to the edge resting on the surface, which lies along a line that connects the two opposite corners

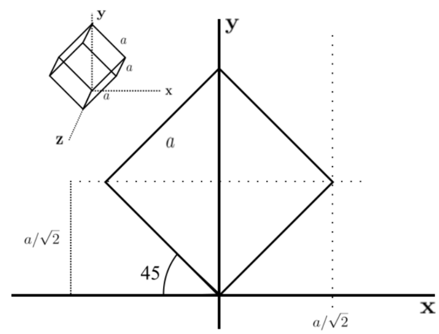


Figure 2: Depiction of the cube on its edge from problem #2, projected to the $x-y$ plane. The inlay shows the three dimensional system.

## 3.1 Moment of Inertia.

The moment of inertia for the cube rotating about one edge can be found using the definition for the components in a continuous mass distribution,

$$I_{ij} = \int_V \rho(\mathbf{r}) \left[ r^2 \delta_{ij} - r_i r_j \right] dV \ , \tag{28}$$

where $r^2$ is the Cartesian distance and $\delta_{ij}$ is the Kronecker delta function, and $\rho(\mathbf{r})$ is the spatial mass density function. For a volume with uniform mass distribution, the density function is the mass per unit volume, in the case of this cube, this is

$$\rho(\mathbf{r}) = \frac{m}{a^3} \ , \tag{29}$$

which is a constant. Note for a square of side length $a$ the distance from a corner to the center is $\sqrt{2}a/2$.

By the parallel axis theorem, the total inertia tensor is the sum of the inertia tensor of a cube about an axis normal to two opposite faces through the center of each face, $\hat{I}_{cube}$, plus the inertia tensor of a point mass $m$ at the center of mass of the cube, rotating about the axis of interest, $\hat{I}_{CoM}$. The inertia tensor for a point mass located at the cube's center of mass, $(x,y,z) = (0, a/\sqrt{2}, 0)$, rotating around the $x$ axis is

$$I_{ij} = m[(x^2 + y^2 + z^2)\delta_{ij} - r_i r_j] \ , \tag{30}$$

where $\delta_{ij}$ is the Kroneker delta function and $r_i \in \{x, y, x\}$. So the diagonal components and off-diagonal components are

$$I_{ii} = m[r_j^2 + r_k^2] \tag{31}$$

$$I_{jk} = -m r_j r_k \ , \tag{32}$$

but since the only nonzero $r_i$ is $y$, the components $I_{jk} = 0$ for $j \neq k$, and $I_{yy} = 0$. The other components are simply $my^2$. Therefore the inertia tensor for the center of mass of the cube rotating about the $x$ axis is

$$\hat{I}_{CoM} = \frac{1}{2} m a^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

The inertia tensor for the cube rotating about an axis that passes through its center of mass and is normal to two opposite faces is a more complex calculation. Set the origin to be the center of mass of the cube, such that the faces of the cube are at $\pm a/2$ in each coordinate direction. Using this coordinate system and the definition of the components of inertia tensor for a continuous mass distribution, Equation 28, the components are given by

$$I_{ij} = \frac{m}{a^3} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \left[ (x^2 + y^2 + z^2)\delta_{ij} - r_i r_j \right] dx\ dy\ dz \ . \tag{34}$$

The first component is

$$I_{11} = \frac{m}{a^3} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \left[ (x^2 + y^2 + z^2 - x^2 \right] dx\ dy\ dz \tag{35}$$

$$= \frac{m}{a^3} \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} y^2 dy \int_{-a/2}^{a/2} z^2 dz = \frac{m}{a^3} \frac{a^5}{6} \ , \tag{36}$$

which by permutation is also equal to $I_{22}$ and $I_{33}$, with for a value of $(1/6)ma^2$. The off-diagonal components are

$$I_{ij} = \frac{m}{a^3} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \left[ -r_i r_j \right] dx\ dy\ dz \ , \tag{37}$$

which are all zero becuase symmetric integrals of an odd function are zero, and the integrands are linear in two of the coordinates for each component. Therefore, the inertia tensor of a cube about an axis passing through its center of mass, normal to two faces is

$$\hat{I}_{cube} = \frac{1}{6} ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ . \tag{38}$$

Which makes the total inertia tensor of a cube rotating about the edge aligned with the $x$ axis

$$\hat{I} = \frac{1}{2} ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{6} ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{6} ma^2 \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \ . \tag{39}$$

Given the complete moment of inertia tensor, the moment of inertia tensor can be found by computing

$$I = \hat{\mathbf{n}}^\dagger \hat{I} \hat{\mathbf{n}} \ , \tag{40}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing along the axis of rotation, and the dagger denotes the transpose. In this case $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, so the moment of inertia about the $x$-axis is

$$I = \frac{1}{6} ma^2 [1, 0, 0] \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{6} ma^2 [4, 0, 0] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{2}{3} ma^2 \ . \tag{41}$$

### 3.2 Angular Velocity.

Assume the edge of the die does not slide on the table. You balance the die on its edge, and eventually it topples. Given that the moment of inertia scalar about the corner of the cube is $I = (2/3)ma^2$, the energy the cube has just before it hits the table (where it has no potential) is

$$E_f = \frac{1}{2} I \omega^2 \ . \tag{42}$$

Initially, in the unstable equilibrium position, the cube only has potential energy, equal to all of its mass located at the center of mass, which is a distance of $a/\sqrt{2}$ from the surface it rests on. After it falls, the center of mass is located at a height $a/2$ above the surface. If the surface is defined as the zero potential energy level, then after the die comes to rest, it will still have some potential energy. To correct for this, set the potential energy zero level to be at a height $a/2$ above the surface the die stands on. This means the new distance from the center of mass is located from the zero potential level (when the die is balanced on an edge) is $(a/\sqrt{2}) - (a/2)$, so the initial energy (purely potential) is given by

$$E_i = mga\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) \ , \tag{43}$$

so by conservation of energy, the angular velocity is

$$\omega = \sqrt{\frac{2mga}{I}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)} = \sqrt{\frac{mga}{I}(\sqrt{2}-1)} = \sqrt{\frac{3g}{2a}(\sqrt{2}-1)} \tag{44}$$

# 4 Problem #4: A Conical Top.

Consider a conical top as a uniform cone of mass $M$, maximum radius $R$ and height $h$. Due to the uniform mass distribution, the density function of the cone is

$$\rho(\mathbf{r}) = \frac{M}{\pi R^3 h/3} \; . \tag{45}$$

## 4.1 Upright Position.

Consider the top when it is spinning on its tip in an upright position, with the $\hat{z}$-axis along the symmetry axis of the cone. Set the origin to be the tip of the cone. The components of the inertia tensor for a continuous mass distribution are defined by

$$I_{ij} = \int_V \rho(\mathbf{r}) \left[ r^2 \delta_{ij} - r_i r_j \right] dV \; , \tag{46}$$

with $r_i \in \{x, y, z\}$. The transformation to cylindrical coordinates is

$$x = s\cos\varphi \qquad y = s\sin\varphi \qquad z = z \; , \tag{47}$$

with a volume element is given by $dV = s\ ds\ d\varphi\ dz$. The Cartesian distance is $r^2 = x^2 + y^2 + z^2$, so the inertia tensor components are

$$I_{ij} = \rho \int\int\int \left[ (x^2 + y^2 + z^2)\delta_{ij} - r_i r_j \right] dx\ dy\ dz \; , \tag{48}$$

again with $r_i \in \{x, y, z\}$. This can be changed to cylindrical coordinates by the transformations above. The limits of integration for $\varphi$ and $z$ are straightforward, going over the entire range allowed to the variable. The $s$ limits of integration are constrained by the $z$ coordinate. Consider a point on the surface of the cone, at height $z$. The triangle formed by the symmetry axis of cone, the cone surface, and the top of the cone. The ratio of the two legs of the right triangle is $R/h$, and the ratio of the smaller triangle, with legs $z$ and $s$, has ratio $s/z$, and must be equal to the larger triangle's ratio, because the opening angle of the cone is the same. This limit the $s$ variable to going from $s = 0$ to $s = Rz/h$ for any value of $z$. The diagonal elements can be calculated:

$$I_{11} = \frac{3M}{\pi R^3 h} \int_0^h \int_0^{2\pi} \int_0^{Rz/h} \left[ x^2 + y^2 + z^2 - x^2 \right] s\ ds\ d\varphi\ dz \tag{49}$$

$$= \frac{3M}{\pi R^3 h} \int_0^h \int_0^{2\pi} \int_0^{Rz/h} \left[ s^2 \sin^2\varphi + z^2 \right] s\ ds\ d\varphi\ dz \tag{50}$$

$$= \frac{3M}{\pi R^3 h} \int_0^h \int_0^{2\pi} \left[ \frac{R^4 z^4 \sin^2(\phi)}{4h^4} + \frac{R^2 z^4}{2h^2} \right] d\varphi\ dz \tag{51}$$

$$= \frac{3M}{\pi R^3 h} \int_0^h \left[ \frac{\pi z^4 \left(4h^2 R^2 + R^4\right)}{4h^4} \right] dz \tag{52}$$

$$= \frac{3M}{\pi R^3 h} \left[ \frac{1}{5}\pi h^3 R^2 + \frac{1}{20}\pi h R^4 \right] \tag{53}$$

$$= \frac{3}{20} M \left(4h^2 + R^2\right) \; , \tag{54}$$

note that the integral of sine and cosine squared over a full period are equal so $I_{xx} = I_{yy}$.

$$I_{22} = \frac{3M}{\pi R^3 h} \int_0^h \int_0^{2\pi} \int_0^{Rz/h} \left[ s^2 \cos^2 \varphi + z^2 \right] s \ ds \ d\varphi \ dz = \frac{3}{20} M \left( 4h^2 + R^2 \right) \tag{55}$$

$$I_{33} = \frac{3M}{\pi R^3 h} \int_0^h \int_0^{2\pi} \int_0^{Rz/h} \left[ s^2 \right] s \ ds \ d\varphi \ dz = \frac{3}{10} MR \ , \tag{56}$$

using MATHEMATICA to evaluate the integrals (element $I_{xx}$ was checked by hand). The off diagonal terms are

$$I_{12} = I_{21} = \rho \int \int \int [-xy] \, dx \ dy \ dz = \rho \int_0^h \int_0^{2\pi} \int_0^{Rz/h} \left[ -s^2 \cos\varphi \sin\varphi \right] s \ ds \ d\varphi \ dz \tag{57}$$

$$= \rho \int_0^h \int_0^{2\pi} \frac{R^4 z^4 \sin(\varphi) \cos(\varphi)}{4h^4} \ d\varphi \ dz = 0 \tag{58}$$

$$I_{13} = I_{31} = \rho \int \int \int [-xz] \, dx \ dy \ dz = \rho \int_0^h \int_0^{2\pi} \int_0^{Rz/h} \left[ -s^2 \cos\varphi \right] s \ ds \ d\varphi \ dz = 0 \tag{59}$$

$$I_{23} = I_{32} = \rho \int \int \int [-yz] \, dx \ dy \ dz = \rho \int_0^h \int_0^{2\pi} \int_0^{Rz/h} \left[ -s^2 \sin\varphi \right] s \ ds \ d\varphi \ dz = 0 \ , \tag{60}$$

because the integrals of sine and cosine over one full period is zero. This means the inertia tensor is given by

$$\hat{I} = \begin{bmatrix} \frac{3}{20} M \left( 4h^2 + R^2 \right) & 0 & 0 \\ 0 & \frac{3}{20} M \left( 4h^2 + R^2 \right) & 0 \\ 0 & 0 & \frac{3}{10} MR \end{bmatrix} \ , \tag{61}$$

## 4.2 Angled Position.

Suppose the cone is spun so its point is fixed, but the axis is inclined to the vertical by some constant angle $\alpha$. The top is precessing around the vertical with a period $\tau$. Maintain the same coordinate frame used in the upright position. For this scenario, there are no external forces and therefore no external torques, so the angle of inclination has no effect on the motion of the cone. As derived in class, the relation between the precession angular velocity $\Omega$ and the angular velocity about the symmetry axis $\omega_3$ is

$$\Omega = \frac{I_3 - I_1}{I_1} \omega_3 \ , \tag{62}$$

but given the precession period, the angular velocity about the symmetry axis is

$$\omega_3 = \frac{2\pi}{\tau} \frac{\frac{3}{20} M \left( 4h^2 + R^2 \right)}{\left[ \frac{3}{10} MR \right] - \frac{3}{20} M \left( 4h^2 + R^2 \right)} = \frac{2\pi}{\tau} \frac{\frac{1}{20} (4h^2 + R^2)}{\frac{1}{10} (R - 2h^2 + \frac{1}{2} R^2)} = \frac{\pi}{\tau} \frac{(4h^2 + R^2)}{(R - 2h^2 + \frac{1}{2} R^2)} \tag{63}$$

# 5 Goldstein 5.15.

Consider a flat rigid body in the shape of a $45^o$ right triangle with uniform mass density and total mass $M$. The length of each leg is $a$, so the density is $\sigma = 2M/a^2$. Use a Cartesian coordinate frame with $\hat{x}$ and $\hat{y}$ pointing along the legs of the right triangle, with the origin located at the right angle. Consider three inertia tensors: $\hat{I}_1$ the inertia tensor of the triangle about its center of mass in the $x{-}y$ plane, $\hat{I}_2$ the inertia tensor of the triangle rotating about the origin in the $x-y$ plane, and $\hat{I}_3$ the inertia tensor of a point mass of mass $M$ rotating around the origin in the $x-y$ plane. All rotations are around axes pointing in the $\hat{z}$ direction, thus the parallel axis theorem applies,

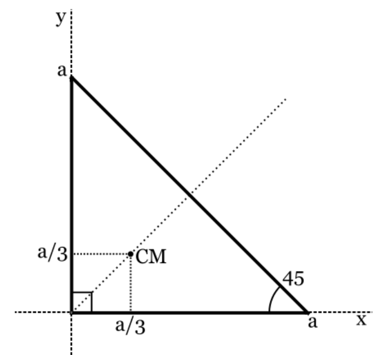


Figure 3: Diagram of geometric plate and coordinate system.

$$\hat{I}_2 = \hat{I}_1 + \hat{I}_3 \ . \tag{64}$$

For a triangle rotating about its center of mass, the moment of inertia of interest is $\hat{I}_1$, which will be called $\hat{I}$ from this point on. For clarity let $\hat{I}_{tri} = \hat{I}_2$, and $\hat{I}_{CM} = \hat{I}_3$. Using this notation, the inertia tensor of the triangle rotating about its center of mass is given by

$$\hat{I} = \hat{I}_{tri} - \hat{I}_{CM} \ , \tag{65}$$

which can be calculated after finding the secondary inertia tensors.

## 5.1 Inertia Tensor for Center of Mass.

The components of the inertia tensor for the point mass of mass $M$ rotating around the origin is given by

$$\hat{I}_{CM(ij)} = M(r^2\delta_{ij} - r_ir_j) \ , \tag{66}$$

for $r_i \in \{x, y, z\}$, with the polar distance $r^2 = x^2 + y^2$. For the rotation of interest, $z = 0$ in all integrals. The center of mass of a triangle is located a distance $a/3$ from each leg, which is a distance $\sqrt{2}a/3$ away from the origin, so $\mathbf{r}_{CM} = (x_{CM}, y_{CM}) = (a/3, a/3)$. The diagonal elements are

$$I_{CM(11)} = M(r^2 - x^2) = My^2 = \frac{Ma}{9} = Mx^2 = I_{CM(22)} \tag{67}$$

$$I_{CM(33)} = M(r^2 - z^2) = M(x^2 + y^2) = \frac{2Ma^2}{9} \ , \tag{68}$$

and the off diagonals,

$$I_{CM(12)} = I_{CM(21)} = M(-xy) = \frac{Ma^2}{9} \tag{69}$$

$$I_{CM(13)} = I_{CM(31)} = M(-xz) = 0 = M(-yz) = I_{CM(23)} = I_{CM(32)} \ , \tag{70}$$

so the inertia tensor is

$$\hat{I}_{CM} = \frac{Ma^2}{9} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} . \tag{71}$$

## 5.2 Inertia Tensor for Triangle About Origin.

The components of the inertia tensor for the triangle about an axis through the origin are given by

$$I_{tri(ij)} = \sigma \int \int [(x^2 + y^2)\delta_{ij} - r_i r_j] dx \; dy , \tag{72}$$

with $r_i \in \{x, y, z\}$. This is still an area integral, not a volume integral because the $z$ dimension of the body is infinitesimal. Therefore the $z$ components are still calculated but in every integral $z$ has the value zero. Note that for any given $x$, the limits of integration for $y$ are 0 and $y(x)$. The line that describes the hypotenuse of the triangle is $y(x) = a - x$. With this in mind the integration can be written as a double integral over $x$ (for clarity the inner integral was written in terms of the variable $s$ instead of $x$):

$$I_{tri(ij)} = \sigma \int_0^a \int_0^{a-x} \left[(s^2 + y^2)\delta_{ij} - r_i r_j\right] ds \; dx . \tag{73}$$

Therefore the first diagonal component is

$$I_{tri(11)} = \sigma \int_0^a \int_0^{a-x} \left[(s^2 + y^2) - s^2\right] ds \; dx = \sigma \int_0^a \int_0^{a-x} (a-s)^2 ds \; dx \tag{74}$$

$$= \sigma \int_0^a \int_0^{a-x} [a^2 + s^2 - 2as] \; ds \; dx = \sigma \int_0^a \left[a^2(a-x) + \frac{(a-x)^3}{3} - a(a-x)^2\right] dx \tag{75}$$

$$= \sigma \int_0^a \left[\frac{1}{3}\left(a^3 - x^3\right)\right] dx = \frac{\sigma}{3}\left(a^3(a) - \frac{a^4}{4}\right) = \frac{1}{4}a^4\sigma , \tag{76}$$

where $\sigma$ is the mass per unit area, $\sigma = M/(a^2/2)$, so this becomes

$$I_{tri(11)} = \frac{1}{4}a^4 \frac{2M}{a^2} = \frac{1}{2}Ma^2 . \tag{77}$$

The same process can be carried out for $\hat{I}_{tri(22)}$ and $\hat{I}_{tri(33)}$ using MATHEMATICA to evaluate the integrals:

$$I_{tri(22)} = \sigma \int_0^a \int_0^{a-x} \left[(s^2 + y^2) - y^2\right] ds \; dx = \sigma \int_0^a \int_0^{a-x} \left[s^2\right] ds \; dx = \frac{\sigma}{3}\int_0^a (a-x)^3 dx \tag{78}$$

$$= \frac{\sigma}{3}\frac{a^4}{4} = \frac{1}{2}Ma^2 \tag{79}$$

$$I_{tri(33)} = \sigma \int_0^a \int_0^{a-x} \left[(s^2 + y^2) - z^2\right] ds \; dx = \sigma \int_0^a \int_0^{a-x} \left[s^2 + (a-s)^2\right] ds \; dx \tag{80}$$

$$= \frac{\sigma}{3}\int_0^a \left[\frac{2a^3}{3} - a^2 x + ax^2 - \frac{2x^3}{3}\right]^3 dx = \frac{\sigma}{3}\frac{a^4}{3} = \frac{2}{9}Ma^2 . \tag{81}$$

Quickly note that all the components of $I_{tri}$ with $i = z \neq j$ are zero because the Kronecker delta function is zero and the products $xz$ and $yz$ are zero because $z = 0$, giving

$$I_{tri(13)} = I_{tri(31)} = I_{tri(23)} = I_{tri(32)} = 0 . \tag{82}$$

Computing the final terms yields

$$I_{tri(12)} = I_{tri(21)} = \sigma \int_0^a \int_0^{a-x} [-sy]\, ds\ dx = \sigma \int_0^a \int_0^{a-x} [-s(a-s)]\, ds\ dx \tag{83}$$

$$= \sigma \int_0^a \left[ -\frac{a^3}{6} + \frac{ax^2}{2} - \frac{x^3}{3} \right] dx = -\sigma \frac{a^4}{12} = -\frac{1}{6} Ma^2 \ , \tag{84}$$

so that the total inertia tensor for the triangle rotating about the origin in the $x-y$ plane is

$$\hat{I}_{tri} = Ma^2 \begin{bmatrix} 1/2 & -1/6 & 0 \\ -1/6 & 1/2 & 0 \\ 0 & 0 & 2/9 \end{bmatrix} \ . \tag{85}$$

## 5.3 Total Inertia Tensor: Triangle Rotating About Center of Mass.

Subtracting the two inertia tensors found above yields the inertia matrix for the regular right triangle rotating about its center of mass in the plane, as indicated in Equation 65

$$\hat{I} = Ma^2 \begin{bmatrix} 1/2 & -1/6 & 0 \\ -1/6 & 1/2 & 0 \\ 0 & 0 & 2/9 \end{bmatrix} - \frac{Ma^2}{9} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{Ma^2}{18} \begin{bmatrix} 7 & -1 & 0 \\ -1 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \tag{86}$$

which can be used to find the principal axes of rotation. Clearly the introduction of the third axis was unnecessary as this can be reduced to a 2 dimensional tensor.

## 5.4 Principal Axes of Rotation.

To calculate the principal axes of rotation, the eigenvalue problem must be solved:

$$0 = \frac{Ma^2}{18} \begin{vmatrix} 7-\lambda & -1 & 0 \\ -1 & 7-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \frac{Ma^2}{18} [-\lambda(\lambda-8)(\lambda-6)] \ , \tag{87}$$

using Mathematica to write out the determinant, so the eigenvalues are $\lambda = 8, 6, 0$. Each eigenvalue has a corresponding eigenvector, which is one of the principal axes of rotation.

### 5.4.1 First Principal Axis, $\lambda = 8$

To find the first principal axis, the eigenvalue problem is rewritten with the selected value of lambda:

$$[\hat{I} - 8I_3]\hat{\mathbf{e}}_1 = 0 = \frac{Ma^2}{18} \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} e_{1x} \\ e_{1y} \\ e_{1z} \end{bmatrix} \tag{88}$$

, where $I_3$ is the $3\times3$ identity matrix. This gives three equations for the three unknown components of the first principal axis vector,

$$0 = -e_{1x} - e_{1y} \tag{89}$$

$$0 = -e_{1x} - e_{1y} \tag{90}$$

$$0 = -8e_{1z} \ , \tag{91}$$

so that, $e_{1z} = 0$, and $e_{1x} = -e_{1y}$. This eigenvector, with a factor to force the vector to have unit norm, becomes the principal axis vector,

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0\ . \end{bmatrix} \tag{92}$$

This method will be repeated for the final two principal axes.

### 5.4.2 Second Principal Axis, $\lambda = 6$

Follow the method described for the first principal axis:

$$[\hat{I} - 6I_3]\hat{\mathbf{e}}_2 = 0 = \frac{Ma^2}{18} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} e_{2x} \\ e_{2y} \\ e_{2z} \end{bmatrix} \tag{93}$$

, which gives three equations for the three unknown components of the first principal axis vector,

$$0 = e_{2x} - e_{2y} \tag{94}$$
$$0 = -e_{2x} + e_{2y} \tag{95}$$
$$0 = -6e_{2z}\ , \tag{96}$$

so that, $e_{2z} = 0$, and $e_{2x} = -e_1 2y$. This eigenvector, with a factor to force the vector to have unit norm, becomes the principal axis vector,

$$\hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0\ . \end{bmatrix} \tag{97}$$

### 5.4.3 Third Principal Axis, $\lambda = 0$

Follow the method described for the first principal axis:

$$[\hat{I} - 0I_3]\hat{\mathbf{e}}_2 = 0 = \frac{Ma^2}{18} \begin{bmatrix} 7 & -1 & 0 \\ -1 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{3x} \\ e_{3y} \\ e_{23z} \end{bmatrix} \tag{98}$$

, which gives three equations for the three unknown components of the first principal axis vector,

$$0 = 7e_{3x} - e_{3y} \tag{99}$$
$$0 = -e_{3x} + 7e_{3y} \tag{100}$$
$$0 = (0)e_{3z}\ , \tag{101}$$

so that, $e_{3z}$ is arbitrary, and $e_{3x} = e_1 3y = 0$. Since the only term in this vector is arbitrary, it is set to 1 to maintain unit norm. So the principal axis vector is

$$\hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1\ . \end{bmatrix} \tag{102}$$

## 6 Goldstein 5.20.

A plane pendulum consists of a uniform rod of length $\ell$ and mass $m$, but negligible thickness. It is suspended in a vertical plane by one end. At the other end, a uniform disk of radius $a$ and mass $M$ is attached at its edge so it can rotate freely in its own plane (about its a point on its circumference, which is the attachment point to the pendulum), which is the vertical plane, as shown in Figure 4. Let $\theta$ be the angle the pendulum makes with the vertical and $\phi$ the angle the circle makes with the horizontal in its rotation about its center of mass. The origin is defined to be the pivot point of the pendulum, with $+\hat{y}$ pointing down the vertical and $+\hat{x}$ going horizontally to the right. The kinetic energy of this system is then

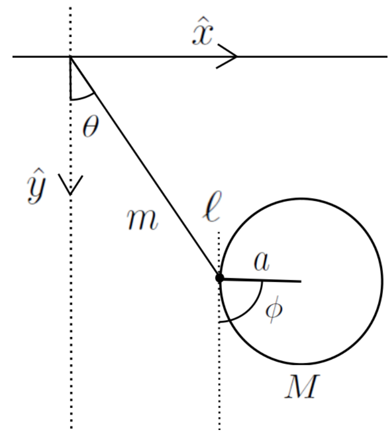

Figure 4: Depiction of the system and coordinates for problem # 6..

$$T = \frac{1}{2} I_{rod}\dot{\theta}^2 + \frac{1}{2} I_{rot}\dot{\phi}^2 + \frac{1}{2} M(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) \ , \tag{103}$$

with $I_{rod}$ as the moment of inertia of the rod about the pivot, $I_{rot}$ as the moment of inertia of the disk rotating about its center of mass, and $\dot{x}_{CM}, \dot{y}_{CM}$ is the $x, y$ position of the center of mass of the disk. The potential energy is due to the gravitational acceleration $g$, and acts on the center of mass of both bodies. The zero level of the potential is defined to be at $y = \ell$, for $y < \ell$ the potential is positive, so that

$$U = U_{rod} + U_{disk} = mgh_{rod} + Mgy_{CM} \ , \tag{104}$$

where $h_{rod}$ is the $y$ position of the rod's center of mass. With these energies the Lagrangian, and thus the equations of motion, can be found. First the moments of inertia must be calculated and the center of mass positions determined. Note that the inertia tensors must be three dimensional, but for each calculation, the value of $z$ is always zero. Additionally, since all rotations are about axes parallel to the $z$ axis, the angular momentum vector looks like $\boldsymbol{\omega} = \{0, 0, \omega\}$, so that the moment of inertia scalar for these three tensors can be found by applying the transformation

$$I = \hat{\mathbf{n}}^\dagger \hat{I} \hat{\mathbf{n}} = \hat{\boldsymbol{\omega}}^\dagger \hat{I} \hat{\boldsymbol{\omega}} = \hat{\mathbf{z}}^\dagger \hat{I} \hat{\mathbf{z}} \ . \tag{105}$$

For this angular momentum, the only component of each inertia tensor that contributes is the $I_{33}$ term:

$$I = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = I_{33} \ . \tag{106}$$

### 6.1 Moment of Inertia for Rod About Origin.

The rod has mass $m$ over a length $\ell$, so that the linear density is $\lambda = m/\ell$. Consider the polar coordinates $r, \varphi$ with the angle measured from the vertical, and the Cartesian coordinate $z$ (which is zero for all calculations). Now a mass element located a distance $r$ from the origin will have coordinates $(x, y) = (r\cos\varphi, r\sin\varphi)$. The inertia tensor components must be integrated over the (radial) length of the rod, so $r = [0, \ell]$. Therefore the components of the inertia tensor for the rod are

$$I_{ij} = \lambda \int_o^\ell [r^2 \delta_{ij} - r_i r_j] dr \tag{107}$$

with $r_i \in (x, y, x)$. The only component that matters is

$$I_{rod} = I_{33} = \lambda \int_o^\ell r^2 dr = \frac{m}{\ell}\frac{\ell^3}{3} = \frac{1}{3} m\ell^2 \ . \tag{108}$$

### 6.2 Moment of Inertia for Disk.

Consider a disk of radius $a$ and mass $M$ oriented in the $x-y$ plane of a Cartesian coordinate system with the center at the origin. In these coordinates a mass element located at $r$ has coordinates $(x, y)$, and all $z$ values are zero. The surface density of this body is $\sigma = M/(\pi a^2)$. For any given $x$ value, the minimum value of $y$ is $-\sqrt{a^2 - x^2}$, and the maximum is $+\sqrt{a^2 - x^2}$, so the components of the inertia tensor are given by

$$I = \sigma \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [(x^2 + y^2)\delta_{ij} - r_i r_j] dy dx \ , \tag{109}$$

with $r_i \in (x, y, x)$. The moment of inertia is then

$$I_{center} = I_{33} = \sigma \int_{-a}^{a} \int_{-(a^2-x^2)^{1/2}}^{(a^2-x^2)^{1/2}} (x^2 + y^2) dy \ dx = \sigma \frac{a^4 \pi}{2} = \frac{1}{2} M a^2 \ , \tag{110}$$

using MATHEMATICA to evaluate the integral. But because the disk rotates about a point on its circumference, the parallel axis theorem states $I_{rot} = I_{center} + Ma^2$, so the total moment of inertia for the disk is $\frac{3}{2} Ma^2$.

### 6.3 Disk Center of Mass Positions.

The disk is attached to the rod by a point on its circumference, therefore the position of the center of mass of the disk is the sum of the position of the end of the rod and the position of the disks center of mass with respect to the end of the rod, or

$$x_{CM} = a \sin\phi + \ell \sin\theta \quad \Rightarrow \quad \dot{x}_{CM} = a\dot{\phi}\cos\phi + \ell\dot{\theta}\cos\theta \tag{111}$$

$$y_{CM} = a \cos\phi + \ell \cos\theta \quad \Rightarrow \quad \dot{y}_{CM} = -a\dot{\phi}\sin\phi - \ell\dot{\theta}\sin\theta \ . \tag{112}$$

The kinetic energy is dependent on the sum of the squares of these derivatives,

$$x_{CM}^2 + y_{CM}^2 = a^2\dot{\phi}^2 + 2a\ell\dot{\theta}\dot{\phi}\cos(\theta - \phi) + \ell^2\dot{\theta}^2 \ . \tag{113}$$

### 6.4 The Lagrangian.

Using the above information and Equations 103 and 104, it is found that

$$T = \frac{1}{6} m\ell^2\dot{\theta}^2 + \frac{3}{4} Ma^2\dot{\phi}^2 + \frac{1}{2} M[a^2\dot{\phi}^2 + 2a\ell\dot{\theta}\dot{\phi}\cos(\theta - \phi) + \ell^2\dot{\theta}^2] \tag{114}$$

$$U = mg(\ell\cos\theta) + Mg(a\cos\phi + \ell\cos\theta) \ . \tag{115}$$

So the Lagrangian is

$$\mathcal{L} = T - U \tag{116}$$

$$= \frac{1}{6}(m + 3M)\ell^2\dot{\theta}^2 + \frac{5}{4} Ma^2\dot{\phi}^2 + \frac{1}{2} M[2a\ell\dot{\theta}\dot{\phi}\cos(\theta - \phi)] - (m + M)g\ell\cos\theta - Mga\cos\phi \ . \tag{117}$$