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Northwestern University PHYS 445, General Relativity
Gravity: An Introduction to Einstein's General Relativity - Hartle
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Contents

1 Hartle 4.9: The Twin Paradox.

Consider twins, Joe and Ed. Joe goes off in a straight line traveling at a speed of $\frac{24}{25}c$ for 7 years as measured on *his* clock, then reverses and returns at half the speed. Ed remains at home. Make a spacetime diagram showing the motion of Joe and Ed from Ed's point of view. When they return, what is the difference in ages between Joe and Ed?

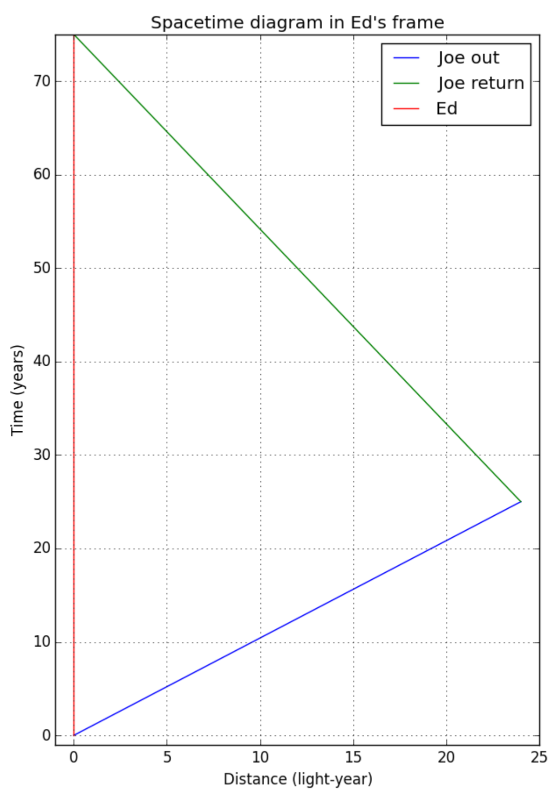
Joe's speed of $\frac{24}{25}c$ is measured with respect to Ed's inertial reference frame. The travel time of $\mathrm{d}\tau = 7$ years is measured in Joe's frame, and is thus the proper time. The time elapsed over the outward trip in Ed's frame is thus

$$\mathrm{d}t = \gamma \mathrm{d}\tau \ , \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \left(\frac{24}{25}\right)^2}} = 3.571 \ , \tag{1}$$

which evaluates to 25 years. On the return trip, Joe moves at $\frac{12}{25}c$, and the trip takes $\mathrm{d}t = 50$ years in Ed's frame. So the time for the return trip, in Joe's frame, is

$$\mathrm{d}t = \gamma \mathrm{d}\tau \ , \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - \left(\frac{12}{25}\right)^2}} = 1.14 \ , \tag{2}$$

which evaluates to $\mathrm{d}\tau = 43.86$ years. Therefore the trip took 51 years measured by Joe and 75 years as measured by Ed, so Ed is 24 years older than Joe upon Joe's return home.



2 Hartle 4.11: Special relativity on a ring.

Alice and Bob are moving in opposite directions around a circular ring of radius R, which is at rest in an inertial frame. Both move with constant speeds V as measured in that frame. Each carries a clock, which they synchronize to zero time at a moment when they are at the same position on the ring. Bob predicts that when next they meet, Alice's clock will read less than his because of time dilation arising because she has been moving with respect to him. Alice predicts that Bob's clock will read less with the same reasoning. They can't both be right. What's wrong with their arguments? What will the clocks really read?

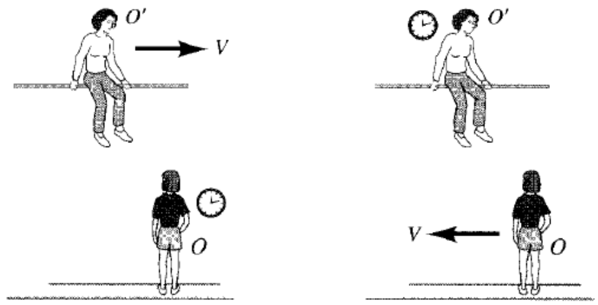
Their error is that they are in rotating reference frames, which are non-inertial due to their acceleration. They have both covered the same distance in the same time, so in the reference frame of the ring, the time until they meet on the other side of the ring is $t = \pi R/V$. Including the Lorentz factor to go from the time in the inertial frame to the proper time of Alice and Bob, we have

$$\tau = \frac{t}{\gamma} = \sqrt{1 - V^2}\frac{\pi R}{V} \ . \tag{3}$$

Note the distance each travels around the ring (in the inertial frame) is only πR because they only make half a revolution before they meet again.

3 Hartle 4.17: Lorentz contraction.

Another derivation of Lorentz contraction Example 4.2 showed how the operation of a model clock was consistent with time dilation. This problem aims at showing how Lorentz contraction is consistent with ideal ways of measuring lengths.



The length of a rod moving with speed V can be determined from the time it takes to move at speed V past a fixed point (left-hand figure). The length of a stationary rod can also be determined by measuring the time it takes a fixed object to move from end to end at speed V (right-hand figure). Taking account of the time dilation between the two times, show that the length of the moving rod determined in this way is Lorentz contracted from its stationary length.

Let the rod have length L (measured when it is at rest). If an observer is moving at speed V relative to the rod, it has a Lorentz factor given by $\gamma = \left(1 - V^2/c^2\right)^{-1/2}$. In the frame of the observer, the time it takes to pass the rod is the proper time, $\mathrm{d}\tau$. Thus, from the rod's point-of-view, the amount of time the rod takes to pass the point is

$$\mathrm{d}t = \gamma \mathrm{d}\tau \ . \tag{4}$$

The length of the rod measured in the stationary frame is L, so the velocity of the observer is

$$V = \frac{L}{\mathrm{d}t} = \frac{L}{\gamma \mathrm{d}\tau} \ , \tag{5}$$

or in another way,

$$V \mathrm{d}\tau = \frac{1}{\gamma} L \ . \tag{6}$$

The quantity $V\mathrm{d}\tau$ is simply the length as measured by the moving observer, L'. Using this, we obtain the desired result:

$$L' = \frac{1}{\gamma} L \ , \tag{7}$$

which says the length of a rod measured by a moving observer is shorter than its rest length by a factor of $\sqrt{1 - V^2/c^2}$.

4 Hartle 5.2: The scalar product.

The scalar product between two three-vectors can be written as

$$\vec{a} \cdot \vec{b} = ab \cos\theta_{ab}$$

where a and b are the lengths of $\vec{a}$ and $\vec{b}$, respectively, and θ_{ab} is the angle between them. Show that an analogous formula holds for two timelike four-vectors **a** and **b**:

$$\mathbf{a} \cdot \mathbf{b} = -ab \cosh\theta_{ab},$$

where $a = (-\mathbf{a} \cdot \mathbf{a})^{1/2}$, $b = (-\mathbf{b} \cdot \mathbf{b})^{1/2}$, and θ_{ab} is the parameter defined in (4.18) that describes the Lorentz boost between the frame where an observer whose world line points along **a** is at rest and the frame where an observer whose world line points along **b** is at rest.

Let observer A have a world line defined by **a**, and be at rest in frame A. Similarly for B. Inner products are Lorentz-invariant quantities, and as such can be evaluated in any frame, with their results holding true in any other inertial frame. In frame A, we can express observer A's world line as

$$\mathbf{a} = (a, 0, 0, 0) \ , \tag{8}$$

which guarantees $a = (-\mathbf{a} \cdot \mathbf{a})^{1/2}$, by construction. In this frame, the scalar product with **b** is

$$\mathbf{a} \cdot \mathbf{b} = a^0 b^0 \eta_{00} = -a b^t_A \ , \tag{9}$$

where b^t_A is the time-component of **b** in frame A. To determine this quantity, we can start in frame B where

$$\mathbf{b} = (b, 0, 0, 0) \ , \tag{10}$$

and boost into frame A. The parameter which defines the boost is θ_{ab}, so using the Lorentz transformations (Hartle 4.18), we see:

$$b^t_A = (\cosh\theta_{ab}) b^t_B \tag{11}$$
$$b^x_A = (-\sinh\theta_{ab}) b^t_B \ , \tag{12}$$

where b^t_B is simply the norm of the vector, b. Using the temporal component of the **b** in the boosted frame, we obtain our result:

$$\mathbf{a} \cdot \mathbf{b} = -ab \cosh\theta_{ab} \ . \tag{13}$$

5 Hartle 5.3: World lines and proper time.

A free particle is moving along the x-axis of an inertial frame with speed $\frac{\mathrm{d}x}{\mathrm{d}t} = V$ passing through the origin at $t = 0$. Express the particle's world line parametrically in terms of V using the proper time τ as the parameter.

The particle's trajectory is easily parameterized in terms of the time measured in the inertial frame: $x(t) = Vt$. The Lorentz factor for the particle is $\gamma = \left(1 - V^2\right)^{-1/2}$, so the proper time for the particle, after time t has passed in the inertial frame is

$$\tau = \frac{t}{\gamma} \ , \tag{14}$$

so the particle's trajectory can be written as

$$x(\tau) = V\gamma\tau \ . \tag{15}$$

Expressing both of these in terms of only V and τ, we have

$$t(\tau) = \tau \left(1 - V^2\right)^{-1/2} \tag{16}$$

$$x(\tau) = \tau V \left(1 - V^2\right)^{-1/2} \ . \tag{17}$$

The uninteresting coordinates are simply:

$$y(\tau) = z(\tau) = 0 \ . \tag{18}$$

6 Hartle 5.4: Four-acceleration.

Work out the components of the four-acceleration vector $\mathbf{a} \equiv \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau}$ in terms of the three-velocity $\vec{V}$ and three-acceleration $\vec{a} = \frac{\mathrm{d}\vec{V}}{\mathrm{d}t}$ to obtain expressions analogous to (5.28). Using this expression and (5.28), verify that $\mathbf{a} \cdot \mathbf{u} = 0$.

The four-velocity is defined to be

$$\mathbf{u} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \ , \tag{19}$$

where the Lorentz factor is $\gamma = \left(1 - |\vec{V}|^2\right)^{-1/2}$. The components of which are

$$u^0 = \gamma \frac{\mathrm{d}x^0}{\mathrm{d}t} = \gamma \tag{20}$$

$$u^i = \gamma \frac{\mathrm{d}x^i}{\mathrm{d}t} = \gamma V^i \ . \tag{21}$$

Gathering the components, we can write the four-velocity as

$$\mathbf{u} = \gamma \left(1, \vec{V}\right) \ . \tag{22}$$

To obtain the four-acceleration, we differentiate with respect to proper time once more:

$$\mathbf{a} \equiv \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \gamma \left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}, \gamma \frac{\mathrm{d}\vec{V}}{\mathrm{d}t} + \vec{V}\frac{\mathrm{d}\gamma}{\mathrm{d}t} \right) \ , \tag{23}$$

We are left with the task of finding $\frac{\mathrm{d}\gamma}{\mathrm{d}t}$, which by the chain rule is

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - \vec{V}(t)\cdot\vec{V}(t)\right)^{-1/2} = -\frac{1}{2}\left(1 - |\vec{V}|^2\right)^{-3/2}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\vec{V}\cdot\vec{V}\right) \tag{24}$$

$$= \frac{1}{2}\gamma^3 \left\{ \frac{\mathrm{d}\vec{V}}{\mathrm{d}t}\cdot\vec{V} + \vec{V}\cdot\frac{\mathrm{d}\vec{V}}{\mathrm{d}t} \right\} = \gamma^3\left(\vec{V}\cdot\vec{a}\right) \ , \tag{25}$$

so that

$$\mathbf{a} = \gamma\left(\gamma^3\left(\vec{V}\cdot\vec{a}\right), \gamma\vec{a} + \gamma^3\left(\vec{V}\cdot\vec{a}\right)\vec{V}\right) = \gamma^4\left(\vec{V}\cdot\vec{a}\right)\left(1, \frac{1}{\gamma^2\left(\vec{V}\cdot\vec{a}\right)}\vec{a} + \vec{V}\right) \ , \tag{26}$$

If we take the inner product of this with the four-velocity, we find:

$$\mathbf{a}\cdot\mathbf{u} = \gamma^5\left(\vec{V}\cdot\vec{a}\right)\left\{-1 + \frac{\vec{V}\cdot\vec{a}}{\gamma^2\left(\vec{V}\cdot\vec{a}\right)} + \vec{V}\cdot\vec{V}\right\} = \gamma^5\left(\vec{V}\cdot\vec{a}\right)\left\{-1 + \gamma^{-2} + |\vec{V}|^2\right\} \tag{27}$$

$$= \gamma^5\left(\vec{V}\cdot\vec{a}\right)\left\{-1 + \gamma^{-2} + \left(1 - \gamma^{-2}\right)\right\} = 0 \ , \tag{28}$$

as required.

7 Hartle 5.6: Trajectory of accelerating particle.

Consider a particle moving along the x-axis whose velocity as a function of time is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{gt}{\sqrt{1+g^2t^2}},$$

where g is constant.

7.a Does the particle's speed ever exceed the speed of light?

No. To show this, we can look at limiting cases. First in the parameter g:

$$\lim_{g\to 0} = 0 \ , \qquad \lim_{g\to\infty} = \frac{t}{|t|} \ , \qquad \lim_{g\to-\infty} = -\frac{t}{|t|} \ . \tag{29}$$

Now if we explore the limits as $t \to \pm\infty$, we see the velocity approaches ± 1 in all three cases. Therefore the speed of the particle never goes beyond $\pm c$. Alternatively, we can square both sides of the given velocity:

$$v^2 = \frac{(gt)^2}{1+(gt)^2} \ , \tag{30}$$

and we can see the right-hand side must be less than one, so taking the square-root:

$$|v| < 1 \ , \tag{31}$$

and the velocity never exceeds the speed of light.

7.b Calculate the components of the particle's four-velocity.

The four-velocity is

$$u^\alpha = \gamma(1, \mathbf{V}) \ , \tag{32}$$

where the Lorentz factor is

$$\gamma = \left[1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right]^{-1/2} = \left[1 - \frac{g^2t^2}{1+g^2t^2}\right]^{-1/2} = \left[1 - \frac{g^2t^2}{1+g^2t^2}\right]^{-1/2} = \sqrt{1+g^2t^2} \ . \tag{33}$$

Thus the components of the four-velocity are

$$u^0 = \sqrt{1+g^2t^2} \tag{34}$$
$$u^1 = gt \tag{35}$$
$$u^2 = u^3 = 0 \ . \tag{36}$$

Which we can show obeys the normalization condition $u^\alpha u_\alpha = -1$:

$$u^\alpha u_\alpha = -\left(\sqrt{1+g^2t^2}\right)^2 + (gt)^2 = -1 \ . \tag{37}$$

7.c Express x and t as a function of the proper time along the trajectory.

Due to the fact that velocity is changing over time, the Lorentz factor is also changing with time. To find the total proper time elapsed at time t, we must integrate the definition of proper time:

$$\mathrm{d}t = \gamma \mathrm{d}\tau \quad \Rightarrow \quad \tau = \int_0^t \gamma(t)^{-1}\mathrm{d}t = \int_0^t \sqrt{1-v^2}\mathrm{d}t = \int_0^t \left(1+g^2t^2\right)^{-1/2}\mathrm{d}t \ . \tag{38}$$

Using integral tables, we can identify this as an inverse hyperbolic sine:

$$\tau = \frac{1}{g}\sinh^{-1}(gt) \ , \tag{39}$$

which we can rearrange to solve for t:

$$t = \frac{1}{g}\sinh(g\tau) \ . \tag{40}$$

As such, the (relevant) components of the four-velocity are

$$u^0 = \sqrt{1+\sinh^2(g\tau)} = \cosh(g\tau) \tag{41}$$

$$u^1 = \sinh(g\tau) \ . \tag{42}$$

To determine the trajectory $x(\tau)$, first we find $x(t)$ which is given by the integral:

$$x(t) - x_0 = \int_0^t \mathrm{d}t\frac{\mathrm{d}x}{\mathrm{d}t} = \int_0^t \mathrm{d}t\frac{gt}{\sqrt{1+g^2t^2}} = \frac{1}{g}\left(\sqrt{1+(gt)^2}-1\right) \ . \tag{43}$$

where x_0 is the position of the particle at time $t=0$. If we now insert τ for t, we find

$$x(\tau) - x_0 = \frac{1}{g}\left(\sqrt{1+\sinh^2(g\tau)}-1\right) = \frac{1}{g}\left(\cosh(g\tau)-1\right) \ . \tag{44}$$

Note we could have gotten the same result by integrating the x-component of the four-velocity wth respect to proper time:

$$x(\tau) - x_0 = \int_0^\tau \mathrm{d}\tau' \sinh(g\tau') = \frac{1}{g}\left(\cosh(g\tau)-1\right) \ . \tag{45}$$

7.d What are the components of the four-force and the three-force acting on the particle?

The four-force is defined to be

$$\mathscr{F}^\alpha \equiv m\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} = m\frac{\mathrm{d}}{\mathrm{d}\tau}u^\alpha \ , \tag{46}$$

the components of which are

$$\mathscr{F}^0 = mg\sinh(g\tau) \tag{47}$$

$$\mathscr{F}^1 = mg\cosh(g\tau) \tag{48}$$

$$\mathscr{F}^2 = \mathscr{F}^3 = 0 \ . \tag{49}$$

The four-force $\mathscr{F}^\alpha$ relates to the three-force $\vec{F}$ by

$$\mathscr{F}^\alpha = \left(\gamma \vec{F} \cdot \vec{v}, \gamma \vec{F}\right) \ , \tag{50}$$

so we obtain the relation

$$\mathscr{F}^1 = \gamma F^x \ . \tag{51}$$

Rearranging and inserting our results, we find:

$$\vec{F} = \frac{1}{\gamma}\mathscr{F}^1 \hat{\mathrm{x}} = \frac{mg\cosh(g\tau)}{\sqrt{1+g^2t^2}}\hat{\mathrm{x}} = \frac{mg\cosh(\sinh^{-1}(gt))}{\sqrt{1+g^2t^2}}\hat{\mathrm{x}} = mg\frac{\sqrt{g^2t^+1}}{\sqrt{1+g^2t^2}}\hat{\mathrm{x}} = mg\hat{\mathrm{x}} \ , \tag{52}$$

where $\hat{\mathrm{x}}$ is the unit three-vector along the x axis.

8 Hartle 5.10: High-energy leptons.

In the LEP particle accelerator at CERN, electrons and positrons travel in opposite directions around the circular ring approximately 10 km in radius at energy of 100 GeV apiece.

8.a How close are these particles to moving at the velocity of light?

The relativistic energy of a particle is given by $E = \gamma m$, so

$$\frac{E}{m} = \left(1 - V^2\right)^{-1/2} \quad \Rightarrow \quad V = \sqrt{1 - \left(\frac{m}{E}\right)^2} \ . \tag{53}$$

Inserting for the mass of the electron (511 keV), we find

$$V = \sqrt{1 - \left(\frac{511 \times 10^3}{100 \times 10^9}\right)^2} = 0.99999999998694389. \tag{54}$$

The particles are moving within 1.4 parts in 10^{11} of the speed of light.

8.b Electrons or positrons can be stored for 2 h. How many turns will an electron or positron make around the ring in this time?

Using $c = 3 \times 10^8$ m/s and $R = 10^4$ m, in 2 hours, the particles travel a distance of

$$d = (2 \times 3600)(Vc) \tag{55}$$

(here V is dimensionless). Therefore the number of turns a particle makes in this time is

$$\frac{2Vc(3600)}{2\pi R} = 3600 \frac{Vc}{\pi(10^4 \text{m})} = 3.4 \times 10^7 \ . \tag{56}$$

Alternatively, we can find the time it takes for a particle to make one complete loop:

$$t = \frac{2\pi R}{Vc} = 2.1 \times 10^{-4} \text{ s} \ . \tag{57}$$

So in two hours, the particle makes

$$\frac{2 \cdot 3600}{t} = 3.4 \times 10^7 \text{ s} \tag{58}$$

turns.

9 Hartle 5.19: Basis vectors.

An observer moves with a constant speed V along the x-axis of an inertial frame. Find the components in that frame of orthonormal basis four-vectors $\{\mathbf{e}_{\hat{\alpha}}\}$ to which the observer can refer observations.

For the first basis vector, we can choose the observer's four velocity:

$$\mathbf{e}_{\hat{0}} = u^{\alpha}_{\text{obs}} = \gamma\,(1, V, 0, 0)\ \ , \tag{59}$$

where the Lorentz factor is $\gamma = \left(1 - V^2\right)^{-1/2}$. The remaining three basis vectors must be chosen so that they are all orthogonal to $\mathbf{e}_{\hat{0}}$ and each other:

$$\mathbf{e}_{\hat{0}} \cdot \mathbf{e}_{\hat{i}} = 0\ . \tag{60}$$

We can trivially satisfy this criterion for two of the basis vectors:

$$\mathbf{e}_{\hat{1}} = (0, 0, 1, 0) \tag{61}$$
$$\mathbf{e}_{\hat{2}} = (0, 0, 0, 1)\ \ , \tag{62}$$

which are simply unit vectors in the $y-$ and $z-$directions. Taking the scalar product of the remaining basis vector and the three previously defined results in the following conditions:

$$\mathbf{e}_{\hat{3}} \cdot \mathbf{e}_{\hat{0}} = 0 \quad \Rightarrow \quad e^t_{\hat{3}} = e^x_{\hat{3}} \frac{e^x_{\hat{0}}}{e^t_{\hat{0}}} = V e^x_{\hat{3}} \tag{63}$$
$$\mathbf{e}_{\hat{3}} \cdot \mathbf{e}_{\hat{1}} = 0 \quad \Rightarrow \quad e^y_{\hat{3}} = 0 \tag{64}$$
$$\mathbf{e}_{\hat{3}} \cdot \mathbf{e}_{\hat{2}} = 0 \quad \Rightarrow \quad e^z_{\hat{3}} = 0\ . \tag{65}$$

If we set $e^x_{\hat{3}} = 1$, we find our final orthogonal basis vector:

$$\mathbf{e}_{\hat{4}} = \gamma\,(V, 1, 0, 0)\ \ , \tag{66}$$

since condition (63) only fixes the first two components up to a common scalar.

10 Hartle 5.23: Tachyons

10.a Argue that a kind of particle that always moves faster than the velocity of light would be consistent with Lorentz invariance in the sense that if its speed is greater than light in one frame, it will be greater than light in all frames. (Such hypothetical particles are called tachyons.)

A tachyon would have a spacelike trajectory (*i.e.*, any two points on it's trajectory are separated by more space than time), so $\mathrm{d}s^2 > 1$. Since $\mathrm{d}s^2$ is Lorentz-invariant, the tachyon would lay on a spacelike trajectory in any inertial reference frame, and thus be superluminal.

10.b Show that the tangent vector to the trajectory of a tachyon is spacelike and can be written $u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}$, where s is the spacelike interval along the trajectory. Show that $\mathbf{u} \cdot \mathbf{u} = 1$.

We cannot use the tachyon's proper time as our affine parameter because there is no inertial reference frame the concept of proper time is well-defined in. Thus, we define

$$u^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\lambda} \ , \tag{67}$$

so that

$$u^\alpha u_\alpha = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x_\alpha}{\mathrm{d}t} = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 (-1 + |\vec{v}|^2) \ , \tag{68}$$

where $\vec{v}$ is the tachyon's velocity. This quantity is guaranteed to be positive for $|\vec{v}| > 1$. Let us now examine the invariant interval:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \quad \Rightarrow \quad \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = -1 + \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\mathrm{d}t^2} = -1 + |\vec{v}|^2 \ . \tag{69}$$

We can see that if we use $\lambda = s$, then

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-2} = \left(-1 + |\vec{v}|^2\right)^{-1} \ , \tag{70}$$

which if we insert into (68) we obtain our result:

$$u^\alpha u_\alpha = \left(-1 + |\vec{v}|^2\right)^{-1} (-1 + |\vec{v}|^2) = 1 \ . \tag{71}$$

10.c Evaluate the components of a tachyons four-velocity u in terms of the three-velocity $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$.

The temporal component of the four-velocity is given by

$$u^0 \equiv \frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\sqrt{|\vec{v}|^2 - 1}} \ , \tag{72}$$

which one could argue is the equivalent of the Lorentz factor for a superluminal particle, and is therefore consistent with the definition of u^0 for a massive subluminal particle. The spatial components are

$$\vec{u} \equiv \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}s} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\vec{v}}{\sqrt{|\vec{v}|^2 - 1}} \ . \tag{73}$$

10.d Define the four-momentum by $\mathbf{p} = m\mathbf{u}$ and find the relation between energy and momentum for a tachyon.

The tachyon's four-momentum is

$$\mathbf{p} = \frac{m}{\sqrt{|\vec{v}|^2 - 1}}(1, \vec{v}) \quad \Rightarrow \quad \mathbf{p} \cdot \mathbf{p} = \left(\frac{m}{\sqrt{|\vec{v}|^2 - 1}} \right)^2 \left(-1 + |\vec{v}|^2 \right) = m^2 \ . \tag{74}$$

For subluminal particles, the relativistic energy is $E^2 = m^2 + |\vec{p}|^2$, so for superluminal particles, we make the substitution $m^2 \to -m^2$. With this substitution, we find

$$E = \pm \sqrt{|\vec{p}|^2 - m^2} \ . \tag{75}$$

10.e Show that there is an inertial frame where the energy of any tachyon is negative.

Suppose in one frame, the tachyon has $E > 0$, and we boost to a frame moving at speed v along the tachyon's trajectory. In this frame, an observer would measure the tachyon's energy to be

$$E' = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}} = -(E, \vec{p}) \cdot \gamma(1, v\hat{x}) = \gamma(E - v|\vec{p}|) \ , \tag{76}$$

where $\gamma = (1 - v^2)^{-1/2}$ and $\hat{x}$ is a unit three-vector pointing along the tachyon's velocity. Assuming the tachyon hass mass $m \geq 0$, from (75), we have $|E| \leq |\vec{p}|$, so for a sufficiently high v the right-hand side of the above can be negative.

10.f Show that if tachyons interact with normal particles, a normal particle could emit a tachyon with total energy and three-momentum being conserved. *Comment*: The result in (f) suggests that a world containing tachyons would be unstable, and there is no evidence for tachyons in nature.

Conservation of three-momentum and energy is simply the conservation of four-momentum. Consider a subluminal massive particle with four momentum $\mathbf{P}$ and mass M which decays to a tachyon (mass m_t) with four-momentum $\mathbf{p}_t$ and some other particle with four momentum $\mathbf{p}$ and mass m. Conservation of four-momentum tells us

$$\mathbf{P} = \mathbf{p} + \mathbf{p_t} \ . \tag{77}$$

Let us work in the frame where the decaying paticle is at rest. In this frame we are left with the conditions

$$M = p^0 + p_t^0 \tag{78}$$

$$0 = \vec{p} + \vec{p_t} \ . \tag{79}$$

Using the condition from conservation of three-momentum, and inserting for energy, we see

$$M = \sqrt{|\vec{p}|^2 + m^2} - \sqrt{|\vec{p}|^2 - m_t^2} \ , \tag{80}$$

where the minus sign between the terms arises due to the fact tachyons can have negative energy.

For $|\vec{p}| < m_t$, the right-hand side of the above is imaginary, and we can exclude that as a possibility. For $|\vec{p}| \to \infty$ the right-hand side goes to zero. The remaining limit we must investigate is $|\vec{p}| = m_t$. In this case, the right-hand side simplifies to

$$\sqrt{m_t^2 + m^2} \ , \tag{81}$$

from which we cannot infer the relative size to mass M. If $\sqrt{m_t^2 + m^2} > M$, then we conclude

$$0 \leq \sqrt{|\vec{p}|^2 + m^2} - \sqrt{|\vec{p}|^2 - m_t^2} \leq \sqrt{m_t^2 + m^2} \ , \tag{82}$$

and there exists a $\vec{p}$ that satisfies (80). However, if $\sqrt{m_t^2 + m^2} < M$, then

$$0 \leq \sqrt{|\vec{p}|^2 + m^2} - \sqrt{|\vec{p}|^2 - m_t^2} < M \ , \tag{83}$$

and there is no solution to (80).

The issue here is the statement of the problem. In the above derivation we have assumed a particle decays to a new particle and a tachyon:

$$A \to A' + T \ . \tag{84}$$

If the question is interpreted as a normal particle emits a tachyon and remains the same type of particle with different energy, we have

$$A \to A + T \ , \tag{85}$$

and we make the substitution $m \to M$, so

$$M = \sqrt{|\vec{p}|^2 + M^2} - \sqrt{|\vec{p}|^2 - m_t^2} \ , \tag{86}$$

which has the same limiting behavior, but guarantees a solution for some $|\vec{p}|$[1]. I would argue this is what Hartle meant by his comment. In this way, a particle of normal matter at rest could decay to a tachyon and gain energy. This would be, for sure, an unstable universe.

Mike Katz secretly eats meat.

[1]The $|\vec{p}|$ which satisifies this is trivially solved for:

$$|\vec{p}| = \frac{m_t\sqrt{m_t^2 + 4M^2}}{2M} \ , \tag{87}$$

where we have selected the positive solution.