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Northwestern University PHYS 445, General Relativity
Gravity: An Introduction to Einstein's General Relativity - Hartle
February 6, 2018

Contents

1 Hartle 8.2: The two-sphere.

In usual spherical coordinates the metric on a two-dimensional sphere is [cf. (Hartle 2.15)]

$$\mathrm{d}S^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \ ,$$

where a is a constant.

1.a Calculate the Cristoffel symbols "by hand".

The Lagrangian of a system is, in general, given by (Hartle 8.10), which for this system is

$$\mathcal{L} = \left\{ -a^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 - a^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 \right\}^{1/2} \ , \tag{1}$$

where λ is an affine parameter. Let us define the "dot" operator such that $\dot{}\ \equiv \mathrm{d}/\mathrm{d}\lambda$, so that

$$\frac{\partial \mathcal{L}}{\partial\dot\theta} = \frac{1}{2\mathcal{L}}\left(-2a^2\dot\theta\right) \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial\dot\phi} = \frac{1}{2\mathcal{L}}\left(-2a^2(\sin^2\theta)\dot\phi\right) \ . \tag{3}$$

The derivatives of the Lagrangian with respect to the coordinates are

$$\frac{\partial \mathcal{L}}{\partial\theta} = \frac{1}{2\mathcal{L}}\left(-2a^2\sin\theta\cos\theta\dot\phi^2\right) \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial\phi} = 0 \ . \tag{5}$$

With these derivatives in hand, we can write down the Euler-Lagrange equations (Hartle 8.9):

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[-\frac{a^2}{\mathcal{L}}\dot\theta\right] = -\frac{a^2}{\mathcal{L}}(\sin\theta)(\cos\theta)\dot\phi^2 \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[-\frac{a^2}{\mathcal{L}}(\sin^2\theta)\dot\phi\right] = 0 \ . \tag{7}$$

With some math that would make a mathemetician cringe, from (Hartle 8.8) we can show

$$\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\tau} \ , \tag{8}$$

where τ is the proper time. Using this, the Euler-Lagrange equations become

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right] = \frac{1}{\mathcal{L}}(\sin\theta)(\cos\theta)\dot\phi^2 \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[(\sin^2\theta)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] = 0 \ . \tag{10}$$

If we multiply both of these equations by a factor of $1/\mathcal{L}$, we see

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} = (\sin\theta)(\cos\theta)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(\sin^2\theta)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \ . \tag{12}$$

From this we can read off the Cristoffel symbols:

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \tag{13}$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta \ , \tag{14}$$

where we have accounted for the factor of two arising from the symmetry of the "downstairs" indeces of the off-diagonal symbols, and the negative sign from their definition, the geodesic equation (Hartle 8.14). All other symbols are zero.

1.b Show that a great circle is a solution of the geodesic equation. (*Hint*: Make use of the freedom to orient the coordinates so the equation of a great circle is simple.)

If the north pole of the two-sphere is defined by $\theta = 0$, then a great circle at $\theta = \pi/2$ has the equation

$$\phi = s/a \ , \tag{15}$$

where s is the distance around the equator, and a is the radius of the two-sphere. The geodesic equation becomes

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^{i}{}_{\phi\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -\frac{1}{a}\Gamma^{i}{}_{\phi\phi}\frac{\mathrm{d}s}{\mathrm{d}\tau} \ , \tag{16}$$

since θ is a constant. The only nonzero Cristoffel symbol with two downstairs ϕ indeces is with $i = \theta$. However, at $\theta = \pi/2$, this Cristoffel symbol become zero, and the right-hand side is zero. Additionally, the left-hand side of the above becomes zero as well becuase θ is a constant, and the great circle satisfies the geodesic equation.

2 Hartle 8.3: The three-dimensional Schwarzschild spacetime.

A thee-dimensional spacetime has the line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \ .$$

2.a Find the explicit Lagrangian for the variational principle for geodesics in this spacetime in these coordinates.

The metric tensor of this sapcetime is

$$g_{\alpha\beta} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 \\ 0 & 0 & r^2 \end{pmatrix} \ , \tag{17}$$

for $\alpha, \beta \in \{t, r, \phi\}$. Thus the Lagrangian is

$$\mathcal{L} = \left\{\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\right\}^{1/2} \ , \tag{18}$$

where λ is an affine parameter.

2.b Using the results of (a) write out the components of the geodesic equation by computing them from the Lagrangian.

Using the same "dot" notation for derivatives with respect to the affine parameter, we have

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{1}{2\mathcal{L}}\left[2\left(1 - \frac{2M}{r}\right)\dot{t}\right] = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{1}{2\mathcal{L}}\left[-2\left(1 - \frac{2M}{r}\right)^{-1}\dot{r}\right] = -\left(1 - \frac{2M}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}\tau} \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2\mathcal{L}}\left[-2r^2\dot{\phi}\right] = -r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \ . \tag{21}$$

Differentiating each side with respect to λ, and dividing by $\mathcal{L}$, turns the above into

$$\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} + \frac{\mathrm{d}t}{\mathrm{d}\tau}\left\{\frac{2M}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} \tag{22}$$

$$\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[-\left(1 - \frac{2M}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right] = -\left(1 - \frac{2M}{r}\right)^{-1}\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} + \frac{\mathrm{d}r}{\mathrm{d}\tau}\left\{\left(1 - \frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} \tag{23}$$

$$\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[-r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] = -r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} - \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\left\{2r\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} \ . \tag{24}$$

Now for the derivatives of the Lagrangian with respect to the coordinates, whicha are all zero except:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{1}{2\mathcal{L}}\left\{\frac{2M}{r^2}\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\dot{r}^2 - 2r\dot{\phi}^2\right\} \ , \tag{25}$$

dividing by $\mathcal{L}$ to be consistent with the above, this becomes

$$\frac{1}{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial r} = \frac{1}{2}\left\{\frac{2M}{r^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - 2r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right\} . \tag{26}$$

With this we can write the Euler-Lagrange equations:

$$\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} - \frac{1}{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial x^\alpha(\lambda)} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}x^\alpha/\mathrm{d}\tau)} = \frac{\partial \mathcal{L}}{\partial x^\alpha(\tau)} . \tag{27}$$

The temporal and azimuthal components of the equation are relatively simple because the right-hand side of the above is zero:

$$0 = \left(1-\frac{2M}{r}\right)\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{\mathrm{d}t}{\mathrm{d}\tau}\left\{\frac{2M}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} \tag{28}$$

$$0 = -r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} - \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\left\{2r\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} . \tag{29}$$

The radial Euler-Lagrange equation is as ugly as Mike Katz eating a hamburger:

$$\begin{aligned} -\left(1-\frac{2M}{r}\right)^{-1}\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + \frac{\mathrm{d}r}{\mathrm{d}\tau}\left\{\left(1-\frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right\} & \\ = \frac{1}{2}\left\{\frac{2M}{r^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - 2r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right\} &, \end{aligned} \tag{30}$$

but with a bit of beautification it can be expressed as

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \left(1-\frac{2M}{r}\right)\frac{M}{r^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1-\frac{2M}{r}\right)r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 . \tag{31}$$

2.c Read off the nonzero Cristoffel symbols for this metric from your results in (b).

The Christoffel symbols are given by

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = -\Gamma^\alpha_{\ \beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} . \tag{32}$$

Rearranging (28) yields

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = -\left(1-\frac{2M}{r}\right)^{-1}\frac{2M}{r^2}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\left(\frac{r}{2M}-1\right)^{-1}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} , \tag{33}$$

and we see

$$\Gamma^t_{\ tr} = \Gamma^t_{\ rt} = \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2} = \frac{1}{2}\left(\frac{r}{2M}-1\right)^{-1} . \tag{34}$$

Rearranging (29) yields

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -2r^{-1}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} , \tag{35}$$

and we see

$$\Gamma^{\phi}{}_{\phi r} = \Gamma^{\phi}{}_{r\phi} = \frac{1}{r} \ . \tag{36}$$

Reading off (31), we find:

$$\Gamma^{r}{}_{rr} = -\left(1 - \frac{2M}{r}\right)^{-1} \frac{M}{r^2} \qquad \Gamma^{r}{}_{tt} = \left(1 - \frac{2M}{r}\right) \frac{M}{r^2} \qquad \Gamma^{r}{}_{\phi\phi} = -r\left(1 - \frac{2M}{r}\right) \ . \tag{37}$$

All other symbols are zero.

3 Hartle 8.4: Rotating Frames.

The line element of flate spacetime in a frame (t, x, y, z) that is rotating with angular velocity Ω about the z-axis of an inertial frame is

$$\mathrm{d}s^2 = -[1 - \Omega^2(x^2 + y^2)]\mathrm{d}t^2 + 2\Omega(y\mathrm{d}x - x\mathrm{d}y)\mathrm{d}t + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ .$$

3.a Verify this by transforming to polar coordinates and checking that the line element is (Hartle 7.4) with the substitution $\phi \to \phi - \Omega t$.

Using the definitions of the Cartesian coordinates in terms of the spherical polar coordinates, we note the following:

$$\mathrm{d}x = \sin\theta\cos\phi\ \mathrm{d}r + r\cos\theta\cos\phi\ \mathrm{d}\theta - r\sin\theta\sin\phi\ \mathrm{d}\phi \tag{38}$$
$$\mathrm{d}y = \sin\theta\sin\phi\ \mathrm{d}r + r\cos\theta\sin\phi\ \mathrm{d}\theta + r\sin\theta\cos\phi\ \mathrm{d}\phi \tag{39}$$
$$\mathrm{d}z = \cos\theta\ \mathrm{d}r - r\sin\theta\ \mathrm{d}\theta \ . \tag{40}$$

If we insert the differential elements and the coordinates into the line element, we find

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,(\mathrm{d}\phi - \Omega\mathrm{d}t)^2 \ , \tag{41}$$

after simplifying all the trigonometric functions. If we compare this to (Hartle 7.4), we see making the desired substitution, such that we have $\mathrm{d}\phi \to \mathrm{d}\phi - \Omega\mathrm{d}t$, yields the above.

3.b Find the geodesic equations for x, y, and z in the rotating frame.

The metric for this spacetime geometry is

$$g_{\alpha\beta} = \begin{pmatrix} -[1-\Omega^2(x^2+y^2)] & \Omega y & -\Omega x & 0 \\ \Omega y & 1 & 0 & 0 \\ -\Omega x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{42}$$

with the Lagrangian given by

$$\mathcal{L} = \sqrt{-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}} \tag{43}$$
$$= \left\{[1-\Omega^2(x^2+y^2)]\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - 2\Omega\left(y\frac{\mathrm{d}x}{\mathrm{d}\lambda} - x\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda} - \left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\right\}^{1/2} . \tag{44}$$

Using our standard "dot" notation, we find:

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{1}{2\mathcal{L}}\left(-2\dot{x} - 2\Omega y\dot{t}\right) = -\left[\frac{\mathrm{d}x}{\mathrm{d}\tau} + \Omega y\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] \tag{45}$$
$$\frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{1}{2\mathcal{L}}\left(-2\dot{y} + 2\Omega x\dot{t}\right) = -\left[\frac{\mathrm{d}y}{\mathrm{d}\tau} - \Omega x\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] \tag{46}$$
$$\frac{\partial\mathcal{L}}{\partial\dot{z}} = \frac{1}{2\mathcal{L}}\left(-2\dot{z}\right) = -\frac{\mathrm{d}z}{\mathrm{d}\tau} \ , \tag{47}$$

and

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2\mathcal{L}}\left(-2\Omega^2 x\dot{t}^2 + 2\Omega\dot{y}\dot{t}\right) \tag{48}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2\mathcal{L}}\left(-2\Omega^2 y\dot{t}^2 - 2\Omega\dot{x}\dot{t}\right) \tag{49}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 0\ . \tag{50}$$

The last line tells us momentum in the z direction is conserved. Multiply (48)-(50) by $1/\mathcal{L}$ to obtain

$$\frac{1}{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial x} = -\left[\Omega^2 x\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \Omega\frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] \tag{51}$$

$$\frac{1}{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial y} = -\left[\Omega^2 y\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \Omega\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] \tag{52}$$

$$\frac{1}{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial z} = 0\ . \tag{53}$$

We can now obtain the Euler-Lagrange equations (where we've divided by a factor of $\mathcal{L}$:

$$\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right] = \frac{1}{\mathcal{L}}\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\ , \tag{54}$$

where $\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\tau}$, so we obtain the geodesic equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\mathrm{d}x}{\mathrm{d}\tau} + \Omega y\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] = \Omega^2 x\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \Omega\frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{55}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\mathrm{d}y}{\mathrm{d}\tau} - \Omega x\frac{\mathrm{d}t}{\mathrm{d}\tau}\right] = \Omega^2 y\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \Omega\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{56}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} = 0\ . \tag{57}$$

We can carry out the derivatives, and rearrange:

$$y\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} - 2\Omega\frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} + \Omega^2 x\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \tag{58}$$

$$-x\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + 2\Omega\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} + \Omega^2 y\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \tag{59}$$

$$0 = \frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2}\ . \tag{60}$$

To obtain the true geodesic equations, we cannot have non-zero quantities on the left-hand side of the above, so we use the fourth Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{1}{2\mathcal{L}}\left(2[1-\Omega^2(x^2+y^2)]\dot{t} - 2\Omega(y\dot{x} - x\dot{y})\right) = [1-\Omega^2(x^2+y^2)]\frac{\mathrm{d}t}{\mathrm{d}\tau} - \Omega y\frac{\mathrm{d}x}{\mathrm{d}\tau} + \Omega x\frac{\mathrm{d}y}{\mathrm{d}\tau}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0\ ,$$

from which we obtain

$$
\begin{aligned}
0 &= \frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \mathcal{L}}{\partial \dot{t}} \\
&= \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} - \Omega^2\left[2x\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} + x^2\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2}\right] - \Omega^2\left[2y\frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}\tau} + y^2\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2}\right] - \Omega\left[\frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} + y\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}\right] + \Omega\left[\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau} + x\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2}\right] \\
&= \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2}\left[1 - \Omega^2(x^2+y^2)\right] - 2\Omega^2\frac{\mathrm{d}t}{\mathrm{d}\tau}\left[x\frac{\mathrm{d}x}{\mathrm{d}\tau} + y\frac{\mathrm{d}y}{\mathrm{d}\tau}\right] + \Omega\left[x\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} - y\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}\right] \ .
\end{aligned}
$$

For the last line to be true, the coefficient of each power of Ω must vanish separately. Since there is only one term proportional to Ω^0, we find

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0 \ , \tag{61}$$

which reduces (58), (59), and (60) to the desired geodesic equations.

3.c Show that in the nonrelativstic limit these reduce to the usual equations of Newtonian mechanics for a free particle in a rotating frame exhibitng the centrifugal force and the Coriolis force.

In the nonrelativistic limit, $\tau \to t$, so the geodesic equations become

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}x}{\mathrm{d}t} + \Omega y\right] = \Omega^2 x - \Omega\frac{\mathrm{d}y}{\mathrm{d}t} \tag{62}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}t} - \Omega x\right] = \Omega^2 y + \Omega\frac{\mathrm{d}x}{\mathrm{d}t} \tag{63}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = 0 \ . \tag{64}$$

The two interesting equations can be expanded:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \Omega^2 x - 2\Omega\frac{\mathrm{d}y}{\mathrm{d}t} \tag{65}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \Omega^2 y + 2\Omega\frac{\mathrm{d}x}{\mathrm{d}t} \ . \tag{66}$$

If we compare this to Newtonian motion, we see the angular velocity vector is $\vec{\Omega} = \Omega\hat{Z}$, so the centrifugal force is

$$\vec{F}_{\text{centrifugal}} = -\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) = \Omega^2(x\hat{x} + y\hat{y}) \ , \tag{67}$$

where $\hat{x}, \hat{y}, \hat{z}$ are unit three-vectors in their respective directions. The Coriolis force is given by

$$\vec{F}_{\text{Coriolis}} = 2\vec{\Omega}\times\vec{v} = 2\Omega\left(\frac{\mathrm{d}x}{\mathrm{d}t}\hat{y} - \frac{\mathrm{d}y}{\mathrm{d}t}\hat{x}\right) \ . \tag{68}$$

Thus Newton tells us

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \left(\Omega^2 x - 2\Omega\frac{\mathrm{d}y}{\mathrm{d}t}\right)\hat{x} + \left(\Omega^2 y + 2\Omega\frac{\mathrm{d}x}{\mathrm{d}t}\right)\hat{y} \ , \tag{69}$$

per unit mass, which agrees with our norelativistic limit.

4 Hartle 8.6: Norm of four-velocity along a geodesic.

Show by direct calculation from the geodesic equation (Hartle 8.15) that the norm of the four-velocity $\mathbf{u} \cdot \mathbf{u}$ is a constant along a geodesic.

In terms of the four-velocity, the geodesic equation is

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -\Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma \ , \tag{70}$$

from which we wish to show

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\delta\epsilon} u^\delta u^\epsilon) = \frac{\mathrm{d}g_{\delta\epsilon}}{\mathrm{d}\tau} u^\delta u^\epsilon + g_{\delta\epsilon} \frac{\mathrm{d}u^\delta}{\mathrm{d}\tau} u^\epsilon + g_{\delta\epsilon} u^\delta \frac{\mathrm{d}u^\epsilon}{\mathrm{d}\tau} = \frac{\mathrm{d}g_{\delta\epsilon}}{\mathrm{d}\tau} u^\delta u^\epsilon + 2 g_{\delta\epsilon} \frac{\mathrm{d}u^\delta}{\mathrm{d}\tau} u^\epsilon \ . \tag{71}$$

Using the geodesic equation, we can express the above as

$$0 = \frac{\mathrm{d}g_{\delta\epsilon}}{\mathrm{d}\tau} u^\delta u^\epsilon - 2 g_{\delta\epsilon} \Gamma^\delta{}_{\beta\gamma} u^\beta u^\gamma u^\epsilon \ , \tag{72}$$

which implies

$$\frac{\mathrm{d}g_{\delta\epsilon}}{\mathrm{d}\tau} u^\delta u^\epsilon = 2 \Gamma_{\epsilon\beta\gamma} u^\beta u^\gamma u^\epsilon \ , \tag{73}$$

where $\Gamma_{\epsilon\beta\gamma}$ is given by Hartle (8.19):

$$\Gamma_{\epsilon\beta\gamma} = \frac{1}{2} \left\{ \frac{\partial g_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial g_{\epsilon\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\epsilon} \right\} \ . \tag{74}$$

Performing the contraction on the right-hand side of (73) yields

$$2\Gamma_{\epsilon\beta\gamma} u^\beta u^\gamma u^\epsilon = \left\{ \frac{\partial g_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial g_{\epsilon\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\epsilon} \right\} u^\beta u^\gamma u^\epsilon \ . \tag{75}$$

In the third term, let us rename the dummy indeces such that $\beta \leftrightarrow \epsilon$:

$$2\Gamma_{\epsilon\beta\gamma} u^\beta u^\gamma u^\epsilon = \frac{\partial g_{\epsilon\beta}}{\partial x^\gamma} u^\beta u^\gamma u^\epsilon + \frac{\partial g_{\epsilon\gamma}}{\partial x^\beta} u^\beta u^\gamma u^\epsilon - \frac{\partial g_{\epsilon\gamma}}{\partial x^\beta} u^\epsilon u^\gamma u^\beta \ , \tag{76}$$

and we see the last two terms are identical (actually all three are because $u^\alpha u^\beta u^\gamma$ is completely symmetric) and they cancel:

$$2\Gamma_{\epsilon\beta\gamma} u^\beta u^\gamma u^\epsilon = \frac{\partial g_{\epsilon\beta}}{\partial x^\gamma} u^\beta u^\gamma u^\epsilon \ . \tag{77}$$

We can take this result and put it back into (73) to obtain

$$\frac{\mathrm{d}g_{\delta\epsilon}}{\mathrm{d}\tau} u^\delta u^\epsilon = \frac{\partial g_{\epsilon\beta}}{\partial x^\gamma} u^\beta u^\gamma u^\epsilon = \frac{\partial g_{\delta\epsilon}}{\partial x^\gamma} u^\gamma u^\delta u^\epsilon \ , \tag{78}$$

where we have renamed the dummy indeces on the right-hand side. If the above is true, it implies $\mathbf{u} \cdot \mathbf{u}$ is always a constant. So we are left with showing

$$\frac{\mathrm{d}g_{\delta\epsilon}}{\mathrm{d}\tau} = \frac{\partial g_{\delta\epsilon}}{\partial x^\gamma} u^\gamma = \frac{\partial g_{\delta\epsilon}}{\partial x^\gamma} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = \frac{\mathrm{d}g_{\delta\epsilon}}{\mathrm{d}\tau} \ , \tag{79}$$

using the four-velocity definition. This derivation shows $\mathbf{u} \cdot \mathbf{u}$ is a constant.

5 Hartle 8.11: Null geodesics in polar coordinates.

Solve for the *null* geodesics in three-dimensional flat spacetime using polar coordinates so the line element is $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2$. Do light rays move on straight lines?

The metric tensor is defined by $g_{00} = -1$, $g_{11} = 1$, and $g_{22} = r^2$, and all other entries are zero. As such, the Lagrangian is

$$\mathcal{L} = \left\{ \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 \right\}^{1/2} , \tag{80}$$

and is independent of the coordinates t and ϕ. This gives rise to the following Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{1}{\mathcal{L}}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right] = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[-r^2\frac{1}{\mathcal{L}}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right] = 0 \tag{81}$$

$$\frac{1}{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[-\frac{1}{\mathcal{L}}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right] = -r\left(\frac{1}{\mathcal{L}}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 . \tag{82}$$

Using (81) we can define the conserved quantities:

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} \equiv \varepsilon \quad \text{and} \quad -r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \equiv -\ell , \tag{83}$$

where ε and ℓ are constants. Photons follow null geodesics and have a four-velocity $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\lambda$, so the norm of a photon's four-velocity is

$$0 = \mathbf{u}\cdot\mathbf{u} = -\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 . \tag{84}$$

The conserved quantities can be inserted into the scalar product to obtain

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \left[\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\right]^{1/2} = \left[\epsilon^2 - \frac{\ell^2}{r^2}\right]^{1/2} . \tag{85}$$

We wish to determine $r(\phi)$ so we are able to compare it to a straight line in polar coordinates. We can determine this quantity from

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\mathrm{d}r/\mathrm{d}\lambda}{\mathrm{d}\phi/\mathrm{d}\lambda} = \frac{1}{\ell/r^2}\left[\epsilon^2 - \frac{\ell^2}{r^2}\right]^{1/2} = r^2\left[\frac{1}{(\ell/\epsilon)^2} - \frac{1}{r^2}\right]^{1/2} , \tag{86}$$

which is of the same form[1] as (Hartle 8.36) and has the solution given by (Hartle 8.38):

$$r\cos(\phi - \phi_0) = \ell/\epsilon . \tag{87}$$

Note that by (Hartle 8.39) this is the equation of a straigt line - therefore null geodesics are still straight lines in three-dimensional flat space.

[1] Take the inverse of each side.

6 More Wormholes!

Explore further the motion of test particles and photons in the wormhole metric (Hartle 7.39). You are free to do this in any way you want (analytically, numerically, graphically, ...). Be creative! For inspiration you may want to read this Wikipedia article: https://en.wikipedia.org/wiki/Ellis_wormhole.

The Ellis wormhole is defined by the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (b^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \ . \tag{88}$$

Consider an equatorial snapshot (t =const and $\theta = \pi/2$); the line element becomes

$$\mathrm{d}s^2 = \mathrm{d}r^2 + (b^2 + r^2)\mathrm{d}\phi^2 \ , \tag{89}$$

which defines a catenoid surface described, in cylindrical coordinates $\{\rho, \phi, z\}$ by the curve

$$\rho(z) = b\cosh(z/b) \ , \tag{90}$$

revolved around the z axis, with $\rho^2 = r^2 + b^2$. For massive particles, this leads to the equation of motion

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{\ell^2}{2(b^2+r^2)} = \frac{\epsilon^2 - 1}{2} \equiv \mathcal{E} \ , \tag{91}$$

where the constants ℓ and ϵ are defined by conservation laws:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \epsilon \quad \text{and} \quad (b^2 + r^2)\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \ell \ . \tag{92}$$

Note, from the definition of the Lorentz factor, (91) looks like a Hamiltonian, and we can identify the term containing ℓ as a potential. Particles with $\mathcal{E} < \ell^2/2b^2$ do not have enough kinetic energy to ever make it over the potential barrier. Particles with $\mathcal{E} < \ell^2/2b^2$ will overcome the barrier and will pass through the wormhole in finite time. Particles with $\mathcal{E} = \ell^2/2b^2$ will reach the top of the potential in infinite time, see figure 1 for a graphical description. We can show this by solving (91) for $\tau(r)$, and integrating

$$\tau = \int_{R_0}^{0} \mathrm{d}r \left\{ 2\mathcal{E} - \frac{\ell^2}{b^2 + r^2} \right\}^{-1/2} \ . \tag{93}$$

What the above says is that for a particle starting at $r = R_0$ attempting to make it to the top of the potential (center of wormhole, $r = 0$) it takes proper time τ. Before we delve into evaluating this integral, lets consider what happens when $\mathcal{E} = \ell^2/(2b^2) \times (1 \pm \delta)$, with $(\delta < 1)$. In this case we can write the proper time integral as

$$\tau = \frac{b}{\ell}\int_{R_0}^{0} \mathrm{d}r \left\{ (1 \pm \delta) - \left(1 + \frac{r^2}{b^2}\right)^{-1} \right\}^{-1/2} \ . \tag{94}$$

The interesting behavior is for $r < b$, so lets consider $r \ll b$ so that we may Taylor expand:

$$\tau = \frac{b}{\ell}\int_{R_0}^{0} \mathrm{d}r \left\{ \pm\delta + \frac{r^2}{b^2} \right\}^{-1/2} \ . \tag{95}$$

If the test particle has the exact right energy such that $\delta = 0$, then $\tau \propto \int_{R_0}^{0} \mathrm{d}r/r$, which diverges. However, even if $\delta = 0.01$, then the trip takes fintie time to reach the top of the potential.

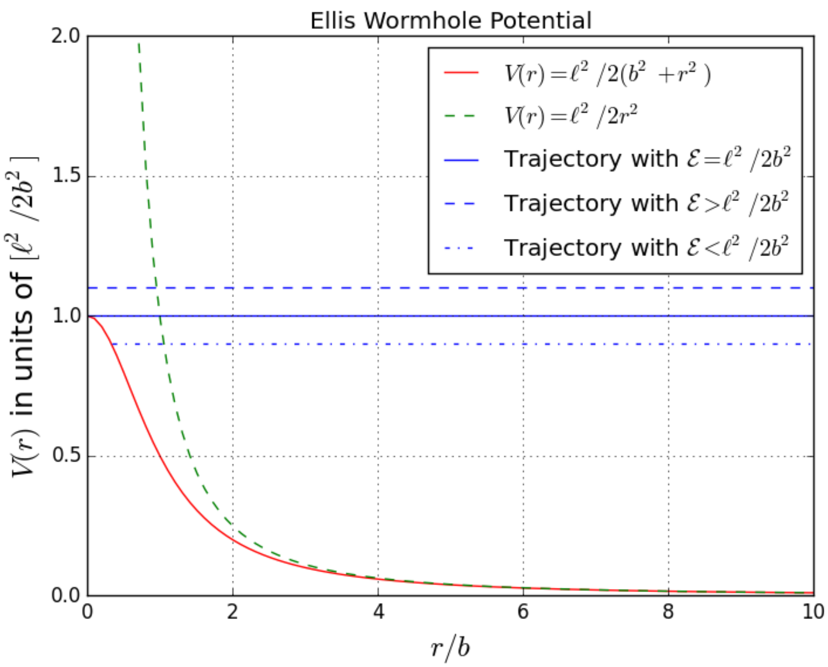

Figure 1: The effective potential induced by the geometry of the Ellis wormhole. The potential is shown in red, and the un-softened ($b = 0$) potential is shown in green. The blue lines are trajectories of massive particles with various energies.