Dylan J. Temples: Solution Set Four

Northwestern University PHYS 445, General Relativity
Gravity: An Introduction to Einstein's General Relativity - Hartle
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Contents

1 Hartle 9.2: Signals of annihilation in a neutron star.

Positrons are produced in the dense plasma surrounding a neutron star, which is accreting material from a binary companion, and electrons and positrons annihilate to produce γ rays. Assuming the neutron star has a mass of $2.5M_\odot$ (solar masses) and a radius of 10 km, at what energy should a distant observer look for the γ rays being emitted from the star by this process? Assume the center of mass of the electron and positron is at rest with respect to the star when they annihilate.

In the process $e^+e^- \to \gamma\gamma$, the total energy of the initial state is $2m_e + 2\epsilon$ in the center of mass, where ϵ is the kinetic energy of either lepton. Still working in the center-momentum frame of the leptons (which is the center-momentum frame for the photons as well) each photon has an energy $m_e + \epsilon$, when they are produced just outside the star. Using the Schwarzschild metric (Hartle 9.9), we see the energy of a photon with four-momentum $\mathbf{p}$, as observed by a stationary observer moving with four-velocity $\mathbf{u}$ is

$$E = -\mathbf{p}\cdot\mathbf{u}_{\text{obs}} = p^0 u^0_{\text{obs}}\left(1 - \frac{2M}{r}\right) = (m_e + \epsilon)\left(1 - \frac{2M}{r}\right)^{1/2} , \tag{1}$$

where we have used the normalization condition $\mathbf{u}_{\text{obs}} \cdot \mathbf{u}_{\text{obs}} = -1$, to determine $u^0_{\text{obs}} = (1 - \frac{2M}{r})^{-1/2}$. Let us insert the constants we have previously set to one so the units don't melt our brains (alternatively use $M_\odot \simeq 1.5$ km):

$$E = (m_e c^2 + \epsilon)\left(1 - \frac{2GM}{c^2 r}\right)^{1/2} , \tag{2}$$

Finally we will assume the electrons and positrons are non-relativistic when they are produced (this is a bad assumption) so that $\epsilon \ll m_e c^2$. We can now plug in values:

$$E = (511 \text{ keV})\left(1 - \frac{2\cdot(6.674\times10^{-20}\text{ km}^3\text{ / kg / s}^2)\cdot 2.5\cdot(2\times10^{30}\text{ kg})}{(3\times10^5\text{ km/s})^2\cdot(10\text{km})}\right)^{1/2} = 259.779 \text{ kev}, \tag{3}$$

so when it has been observed at infinity the photon has lost roughly half of its energy. This energy corresponds to a wavelength of 0.005 nm.

2 Hartle 9.5: Orbits near the top of the effective potential.

Sketch the qualitative behavior of a particle orbit that comes in from infinity with a value of $\mathcal{E}$ exactly equal to the maximum of the effective potential, V_{eff}. How does the picture change if the value of $\mathcal{E}$ is a little bit larger than the maximum or a little bit smaller?

The effective potential (per unit mass) of the Schwarzschild metric is

$$V_{\text{eff}}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} = \frac{1}{2}\left\{\left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) - 1\right\} , \tag{4}$$

where M is the mass of the gravitational source and ℓ is the angular momentum of the test particle with unit mass traversing the spacetime. The effective potential has roots at

$$r = \frac{\ell^2}{4M}\left(1 \pm \sqrt{1 - \left(\frac{4M}{\ell}\right)^2}\right) . \tag{5}$$

The extrema of the potential are given by $\mathrm{d}V_{\text{eff}}(r)/\mathrm{d}r = 0$, which has solutions

$$r_{\text{min}} = \frac{\ell^2}{2M}\left(1 + \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\right) \quad \text{and} \quad r_{\text{max}} = \frac{\ell^2}{2M}\left(1 - \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\right) , \tag{6}$$

note that the labels min and max refer to the value of the potential, not the relative sizes of the r coordinate. Let us define

$$\mathcal{V} = V_{\text{eff}}(r_{\text{max}}) , \tag{7}$$

as the maximum value of the potential (the expression is a mess, like Mike Katz eating barbeque ribs).

Let us consider a point source of mass M producing the potential so for $r > 0$, we are outside the source. For a particle with $\mathcal{E} = \mathcal{V}(1 + \delta)$, where δ is a small parameter, the particle has enough energy to overcome the potential barrier and make it to $r = 0$. This means the particle starts from infinity and spirals into the source in finite time. Contrarily, if the particle has $\mathcal{E} = \mathcal{V}(1 - \delta)$ it cannot make it to the top of the potential and there is a turning point. This means the particle comes in from infinity, reaches a point of closest approach and is flung back out to infinity. The interesting behavior happens when $\mathcal{E} = \mathcal{V}$ exactly because the particle "gets stuck" at the top of the potential. The particle comes in from infinity and sprials inwards, however it never reaches the gravitational source. Instead it gets captured in an unstable circular orbit. See figure 1 for a depiction of the effective potential, and figure 2 for a particle orbit with $\mathcal{E} = \mathcal{V}(1 \pm \delta)$.

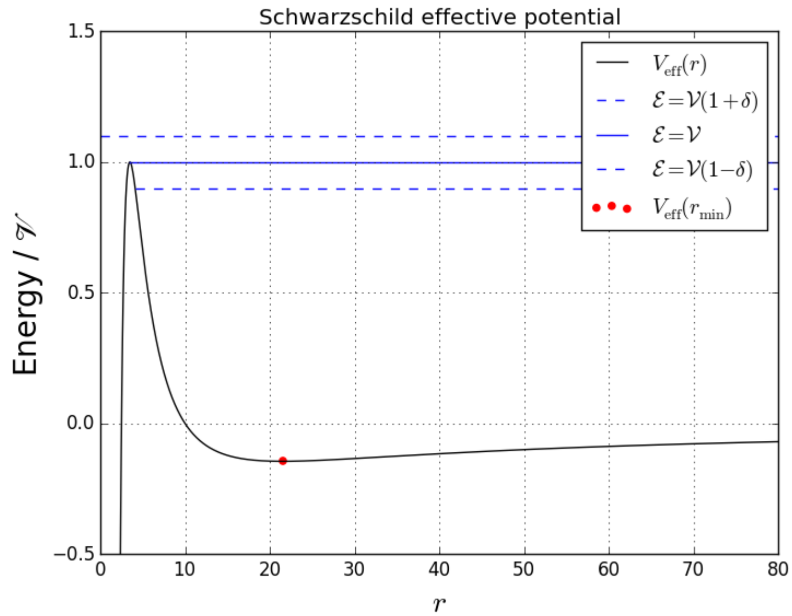

Figure 1: The effective potential as a function fo the Schwarzschild coordinate r, the y-axis is in units of the maximum of the potential, here M has been set to unity, and $\ell = 5.$. The red point indicates the location of the stable circular orbit. The dashed blue lines are particles with energies above and below the peak of the potential. The solid blue line is a particle with energy exactly equal to the peak of the potential.

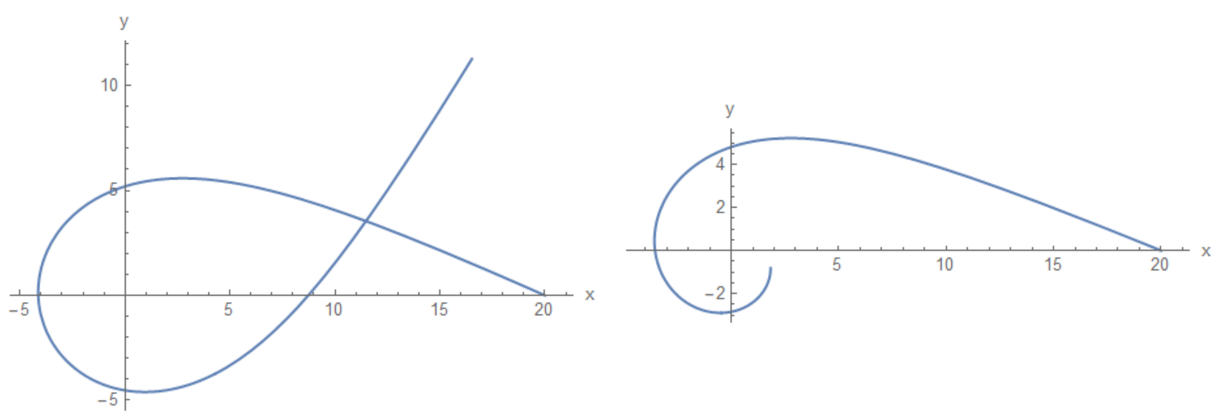

Figure 2: Orbits in $x = r\cos\phi$, $y = r\sin\phi$ for particles in the Schwarzschild geometry with a point source of $M = 1.$. A trajectory is shown for both$\mathcal{E} = \mathcal{V}(1 \pm \delta)$, with $\delta = 0.1$. The "+" trajectory is shown on the left and the "−" on the right (note this trajectory would continue to spiral in but the numerical integration halts at the coordinate singularity of $r = 2M$). Both trajectories have $\ell = 5$. These plots were generated by `schorbits.nb` included on the book website.

3 Hartle 9.8: Orbital periods in the Schwarzschild metric.

A spaceship is moving without power in a circular orbit about a black hole of mass M. (The exterior geometry is the Schwarzschild geometry.) The Schwarzschild radius of the orbit is $7M$.

3.a What is the period of the orbit as measured by an observer at infinity?

The orbial period is easily found if we know the angular velocity of the circular orbit. For general orbits in the Schwarzschild metric, we have

$$\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}\phi/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau} \ . \tag{8}$$

From the conserved quantities, we have

$$\ell \equiv r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \tag{9}$$

$$\varepsilon \equiv \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \ , \tag{10}$$

so we can invert and insert into the first expression:

$$\Omega = \frac{\ell}{\varepsilon}\frac{1}{r^2}\left(1 - \frac{2M}{r}\right) \ , \tag{11}$$

where we have chosen a circular orbit along the equitorial plane ($\theta = \pi/2$, $u^\theta = 0$). A circular orbit ($\mathrm{d}r/\mathrm{d}\tau = 0$) must sit at an extrema of the effective potential, since $7M > r_{\mathrm{ISCO}}$, we will take this orbit to be sitting at the minimum, thus

$$V_{\mathrm{eff}}\left(r = r_{\mathrm{min}}\right) = \frac{\varepsilon^2 - 1}{2} \ , \tag{12}$$

where the effective potential is given by (4), and r_{min} by (6). Trivial algebra yields the condition for circular orbits:

$$\varepsilon^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)\bigg|_{r=r_{\mathrm{min}}} \ , \tag{13}$$

additionally, solving (6) for the angular momentum tells us

$$\ell^2 = \frac{r_{\mathrm{min}}^2}{\frac{r_{\mathrm{min}}}{M} - 3} \ . \tag{14}$$

Let us factor ℓ^2 out of the second term in (13), and then divide both sides by ℓ^2 to obtain

$$\frac{\varepsilon^2}{\ell^2} = \left(1 - \frac{2M}{r}\right)\left(\frac{1}{\ell^2} + \frac{1}{r^2}\right) = \left(1 - \frac{2M}{r}\right)\left(\frac{\frac{r}{M} - 3}{r^2} + \frac{1}{r^2}\right) \tag{15}$$

where we've used (14) on the right-hand side. Note we've simplified our notation such that r now represents the radius of the circular orbit at r_{min}. Some simple algebra yields

$$\frac{\ell}{\varepsilon} = \sqrt{Mr}\left(1 - \frac{2M}{r}\right)^{-1} \ , \tag{16}$$

which we can insert into (11) to obtain

$$\Omega = \sqrt{M/r^3} \ , \tag{17}$$

note this is general for any stable equitorial circular orbit with arbitrary angular momentum. With this in hand, the orbital period measured by a stationary observer at infinity is

$$P_\infty = \frac{2\pi}{\Omega} = 2\pi 7^{3/2} M \ , \tag{18}$$

for $r = 7M$.

3.b What is the period of the orbit as measured by a clock in the spaceship?

To determine the orbital velocity of the ship as measured by a passenger with a wristwatch, we follow the same procedure with the caveat that $\mathrm{d}t/\mathrm{d}\tau$ is not as simple as the stationary observer case. Let us investigate the normalization condition for the four-velocity of the ship on the equatorial circular orbit:

$$-1 = \mathbf{u} \cdot \mathbf{u} = -\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \ . \tag{19}$$

However from (8) we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \Omega \frac{\mathrm{d}t}{\mathrm{d}\tau} \ , \tag{20}$$

so that

$$1 = \left(1 - \frac{2M}{r} - r^2\Omega^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \ . \tag{21}$$

Using our result for the orbital velocity as measured at infinity we see

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{3M}{r}\right)^{-1/2} = \sqrt{\frac{7}{4}} \ , \tag{22}$$

inserting for the radius of the circular orbit. Now using (20) we obtain our result

$$P_{\text{orbit}} = \frac{2\pi}{\mathrm{d}\phi/\mathrm{d}\tau} = \frac{2\pi}{\Omega}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1} = P_\infty \frac{2}{7^{1/2}} = 28\pi M \ , \tag{23}$$

which is shorter than the orbital period measured at infinity.

4 Hartle 9.9: Orbital velocity and Schwarzschild radius.

Find the relation between the rate of change of the angular position of a particle in a circular orbit with respect to proper time and the Schwarzschild radius of the orbit. Compare with (Hartle 9.46).

Given a Schwarzschild radius R for an equitorial, stable, circular orbit, we have from the conservation laws:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\ell}{R^2} \ . \tag{24}$$

We can use the result from the previous problem given in (14) to write this as

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{R}{\sqrt{\frac{R}{M} - 3}} \frac{1}{R^2} = \sqrt{\frac{M}{R^3}} \left(1 - \frac{3M}{R}\right)^{-1/2} \ , \tag{25}$$

but from the last problem we can identify the last factor as

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sqrt{\frac{M}{R^3}} \frac{\mathrm{d}t}{\mathrm{d}\tau} \ . \tag{26}$$

This is the orbital velocity of the circular orbit as measured by the orbiter. Note this is larger than (Hartle 9.46) both because of time dilation and being deeper in the gravitational potential. This problem is almost identical to the previous.

5 Hartle 9.12: Comet and relativstic star.

A comet starts at infinity, goes around a relativistic star of mass M and goes out to infinity. The impact parameter at infinity is b. The Schwarzschild radius of closest approach is R. What is the speed of the comet at closest approach as measured by a stationary observer at that point?

Consider a stationary observer at Schwarzschild radius R. Noting that $\mathbf{u}_{\text{obs}} \cdot \mathbf{u}_{\text{obs}} = -1$, and that none of the spatial coordinates are changing with proper time, its four-velocity is

$$\mathbf{u}_{\text{obs}} = \left(\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0 \right) . \tag{27}$$

Thus if the comet has four-velocity $\mathbf{u}(\tau)$ and mass m, then its energy as measured by the stationary observer is

$$E = -m\mathbf{u}(\tau) \cdot \mathbf{u}_{\text{obs}} = \left(1 - \frac{2M}{R}\right)^{1/2} m u^t \tag{28}$$

At the moment of closest approach the radial coordinate is not changing, so the relevant spatial components of the four-velocity are

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0 \quad \text{and} \quad \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\ell}{R^2} , \tag{29}$$

if we define our coordinates so we are looking at an equitorial plane $\theta = \pi/2$. Note we've introduced ℓ as the conserved angular momentum. To find the remaining component of the four-velocity we exploit $\mathbf{u} \cdot \mathbf{u} = -1$:

$$-1 = -\left(1 - \frac{2M}{R}\right) \left(u^t\right)^2 + R^2 \left(\frac{\ell}{R^2}\right)^2 \quad \Rightarrow \quad u^t = \sqrt{\frac{1 + \frac{\ell^2}{R^2}}{1 - \frac{2M}{R}}} . \tag{30}$$

Using the definition of relativistic energy, we can combine the above and (28) to find

$$\frac{m}{\sqrt{1 - v^2}} = E = \left(1 - \frac{2M}{R}\right)^{1/2} m \sqrt{\frac{1 + \frac{\ell^2}{R^2}}{1 - \frac{2M}{R}}} \quad \Rightarrow \quad \left(1 - v^2\right)^{-1/2} = \left(1 + \frac{\ell^2}{R^2}\right)^{1/2} , \tag{31}$$

where v is the velocity of the comet at its closest approach, R. After a pinch of algebra, this becomes

$$v = \frac{\ell/R}{\sqrt{1 + (\ell/R)^2}} = \left(\frac{R^2}{\ell^2} + 1\right)^{-1/2} . \tag{32}$$

Now we are left with relating the conserved quantities ε and ℓ to the impact parameter b. The relationship between energy per unit mass ε and angular momentum is determined by the turning point condition. At the turning point $r = R$, the energy of the particle is equal to the effective potential ($\mathrm{d}r/\mathrm{d}\tau = 0$). Thus, the radial equation can be simplified to illustrate the turning point condition:

$$\frac{\varepsilon^2 - 1}{2} = -\frac{M}{R} + \frac{\ell^2}{2R^2} - \frac{M\ell^2}{R^3} \tag{33}$$

The impact parameter is defined at infinity, so $b/r \ll 1$, thus $b/r = \tan\phi \sim \phi$. With this, we can write the angular momentum as

$$\ell = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} = -br^2 \left(\frac{1}{r^2} \frac{\mathrm{d}r}{\mathrm{d}\tau} \right) = b \left| \frac{\mathrm{d}r}{\mathrm{d}\tau} \right| \ , \tag{34}$$

since the radial coordinate is decreasing. For all r, the radial equation is

$$\frac{\varepsilon^2 - 1}{2} = \frac{1}{2} \left(\frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 + V_{\text{eff}}(r) \ , \tag{35}$$

but for large r, $V_{\text{eff}} \to 0$ ($\sim 1/r$), so we see

$$\left| \frac{\mathrm{d}r}{\mathrm{d}t} \right| = \sqrt{\varepsilon^2 - 1} \quad \Rightarrow \quad \ell = b\sqrt{\varepsilon^2 - 1} \ . \tag{36}$$

Using (36) to insert for $\varepsilon^2 - 1$ in (33), we find

$$\left(\frac{R}{\ell} \right)^2 = \frac{R}{2M} - \frac{R^3}{2Mb^2} - 1 = \frac{b^2 R - 2b^2 M - R^3}{2b^2 M} \ , \tag{37}$$

which we can insert into (32) to obtain

$$v = \left(\frac{R}{2M} - \frac{R^3}{2Mb^2} \right)^{-1/2} = b\sqrt{\frac{2M}{R}} \left(b^2 - R^2 \right)^{-1/2} = \sqrt{\frac{2M}{R}} \left(1 - \frac{R^2}{b^2} \right)^{-1/2} \ . \tag{38}$$