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Northwestern University PHYS 445, General Relativity
Gravity: An Introduction to Einstein's General Relativity - Hartle
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Contents

1 Hartle 12.4: Eddington-Finkelstein coordinates for Schwarzschild-like metric.

Consider the spacetime specified by the line element

$$\mathrm{d}s^2 = -\left(1-\frac{M}{r}\right)^2 \mathrm{d}t^2 + \left(1-\frac{M}{r}\right)^{-2} \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \ . \tag{1}$$

Except for $r = M$, the coordinate t is always timelike and the coordinate r is spacelike.

1.a Find a transformation to new coordinates (v, r, θ, ϕ) analogous to (Hartle 12.1) that sets $g_{rr} = 0$ and shows that the geometry is not singular at $r = M$.

Inspired by the Eddington-Finkelstein coordinates of the Schwarzschild geometry, we define the zero-component coordinate v, by

$$t = v - f(r) \ , \tag{2}$$

where $f(r)$ is a function to be determined by the conditions on the metric tensor. We can insert this into the line element to obtain

$$\mathrm{d}s^2 = -\left(1-\frac{M}{r}\right)^2 \left(\mathrm{d}v^2 + \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 \mathrm{d}r^2 - 2\frac{\mathrm{d}f}{\mathrm{d}r}\mathrm{d}r\mathrm{d}v\right) + \left(1-\frac{M}{r}\right)^{-2} \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \ . \tag{3}$$

We desire the metric tensor to have $g_{rr} = 0$ in exchange for acquiring off-diagonal terms $g_{rv} = g_{vr}$, so we have the condition

$$0 = \left\{\left(1-\frac{M}{r}\right)^{-2} - \left(1-\frac{M}{r}\right)^2 \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2\right\} \mathrm{d}r^2 = \left(1-\frac{M}{r}\right)^2 \left\{\left(1-\frac{M}{r}\right)^{-4} - \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2\right\} \mathrm{d}r^2 \tag{4}$$

from which we see

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \pm\left(1-\frac{M}{r}\right)^{-2} \ , \tag{5}$$

we are free to pick the sign since this quantity only appears squared; we will select the positive solution. Performing the integration yields

$$f(r) = r + 2M\log|r-M| - \frac{M^2}{r-M} \quad \Rightarrow \quad v = t + r + 2M\log|r-M| - \frac{M^2}{r-M} \ , \tag{6}$$

plus an arbitrary constant. We can select this constant such that $v = 0$ at $t = r = 0$, so

$$t = v - (r - M - 2M\log M) - 2M\log|r-M| + \frac{M^2}{r-M} \tag{7}$$

With this transformation, the line element becomes

$$\mathrm{d}s^2 = -\left(1-\frac{M}{r}\right)^2 \mathrm{d}v^2 + 2\mathrm{d}r\mathrm{d}v + r^2\mathrm{d}\Omega^2 \ , \tag{8}$$

which is well-behaved for $r = M$.

1.b Sketch a $(\tilde{t}, r)$ diagram analogous to Hartle Figure 12.2 showing the world lines of ingoing and outgoing light rays and light cones.

Let us define a new time given by

$$\tilde{t} \equiv v - r = t + 2M \log|r - M| - \frac{M^2}{r - M} - (M + 2M \log M) \ . \tag{9}$$

Ignoring the angular dependence, light rays obey the condition

$$0 = \mathrm{d}v \left[-\left(1 - \frac{M}{r}\right)^2 \mathrm{d}v + 2\mathrm{d}r \right] \ , \tag{10}$$

implying that light rays follow trajectories with

$$\frac{\mathrm{d}v}{\mathrm{d}r} = 0 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2}{\left(1 - \frac{M}{r}\right)^2} \tag{11}$$

Using our new time, this is

$$\frac{\mathrm{d}(\tilde{t} + r)}{\mathrm{d}r} = 0 \quad \text{and} \quad \frac{\mathrm{d}(\tilde{t} + r)}{\mathrm{d}r} = \frac{2}{\left(1 - \frac{M}{r}\right)^2} \tag{12}$$

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}r} = -1 \quad \text{and} \quad \frac{\mathrm{d}\tilde{t}}{\mathrm{d}r} = \frac{2}{\left(1 - \frac{M}{r}\right)^2} - 1 \ . \tag{13}$$

The light cones defined by these slopes are shown in figure 1 for representative values of r/M.

1.c Is this the geometry of a black hole?

In a $(\tilde{t}, r)$ spacetime diagram, the ingoing leg of local light cones is at $45°$, for all points in space. However, the slope of the outgoing leg of local light cones increase as one approaches $r = M$ from the outside ($r > M$). At $r = M$ the outoging leg is a vertical line, and for $r < M$ the slope increases in the negative direction as one approaches $r = M$. This is a black hole; nothing at any point with $r < M$ can escape to $r > m$. Once an object passes $R = M$ it can never move in a direction of increasing r.

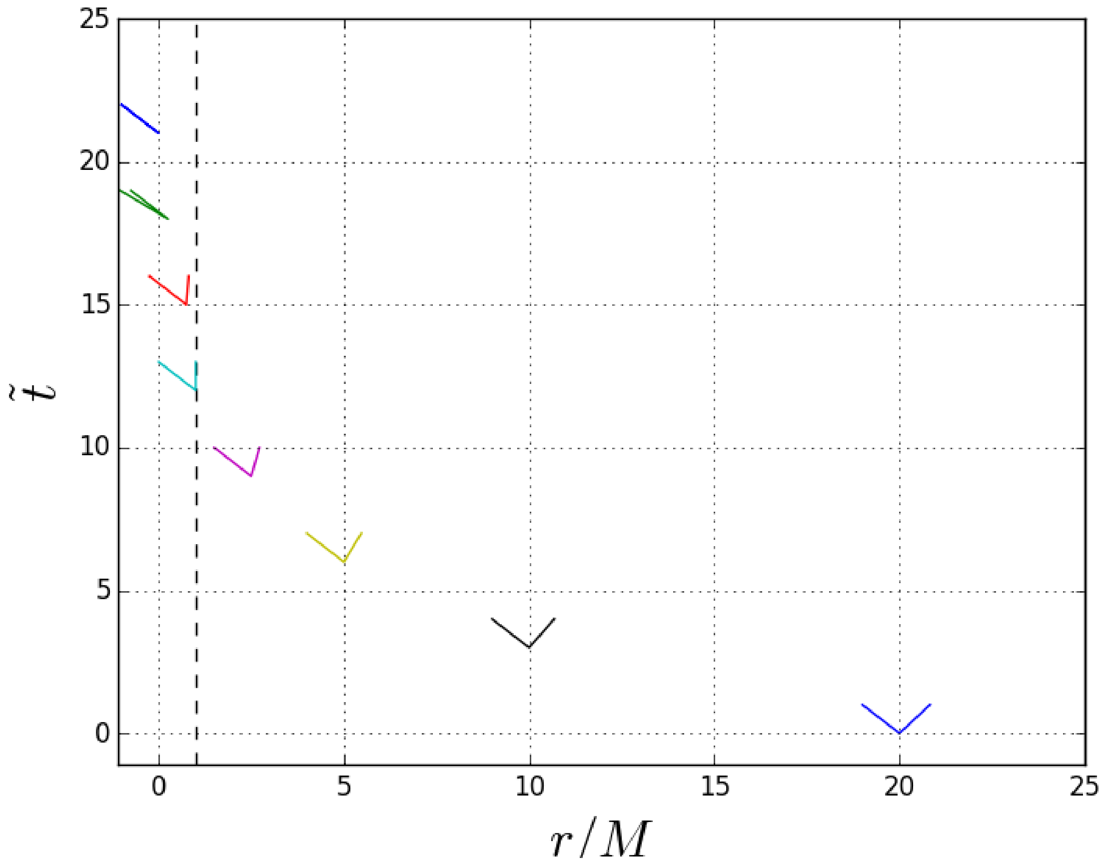

Figure 1: A $(\tilde{t}, r)$ spacetime diagram for the metric specified by (1). Note that in the region $r < M$ there is no motion possible along a trajectory of increasing r, and thus is the spacetime around a black hole.

2 Hartle 12.12: Normal vector of a null surface.

Check that the normal vector to the horizon three-surface of a Schwarzschild black hole is a null vector.

Since we are investigating phenomena at the horizion, we must use coordinates which are non-singular at $r = 2M$; we will use Eddington-Finkelstein coordinates (v, r, θ, ϕ). Following the treatment in Hartle section 7.9, we have three tangent four-vectors t_i^α, where α is a Lorentz index and $i \in (1, 2, 3)$ is a label. Additionally, we have a normal four-vector n^α such that

$$0 = g_{\alpha\beta} t_i^\alpha n^\beta \ , \tag{14}$$

for all i. The horizon three-surface exists at constant r, so a normal vector will have no component in r, thus the tanget vectors are in the v, θ, and ϕ directions:

$$t_1^\alpha = (1, 0, 0, 0) \qquad t_2^\alpha = (0, 0, 1, 0) \qquad t_2^\alpha = (0, 0, 1, 0) \ . \tag{15}$$

Satisfying orthognoality with the final two tangent vectors is trivial, as long as n^α has no components in the θ and ϕ directions. We are left to enforce

$$0 = g_{\alpha\beta} t_1^\alpha n^\beta = g_{v\beta} n^\beta = g_{vv} n^v + g_{vr} n^r = -\left(1 - \frac{2M}{r}\right) n^v + n^r \ , \tag{16}$$

but at the horizon, $r = 2M$, so the first term vanishes, and thus $n_r = 0$, so the normal vector only has a v component. The norm of n^α determines if it is null, so

$$g_{\alpha\beta} n^\alpha n^\beta = g_{vv} (n^v)^2 = -\left(1 - \frac{2M}{r}\right) (n^v)^2 \ , \tag{17}$$

but again on the horizon, $r = 2M$ and the right-hand side vanishes showing that n^α is indeed a null vector.

3 Hartle 12.13: Light transmission and black holes.

3.a An observer falls feet first into a Schwarzschild black hole looking down at her feet. Is there ever a moment when she cannot see her feet? For instance, can she see her feet when her head is crossing the horizon? If so, what radius does she see them at? Does she ever see her feet hit the singularity at $r = 0$ assuming she remains intact until her head reaches that radius? Analyze these questions with an Eddington-Finkelstein or Kruskal diagram.

In the following I will refer to $\tilde{t}$ as "time". Consider the Eddington-Finkelstein diagram given in Hartle Figure 12.2, where lines of constant time are horizontal. The observer falls radially inwards feet-first, so her feet are always at a smaller radius, for the same time. All points on the world line of her head are connected to the world line of her feet by the null geodesics, so she can always see her feet. Consider the moment where the world line of her head crosses the horizon, at time $\tilde{t}_0$. Here the null geodesic is a vertical line. She can see her feet here, but they appear to be at the same radius as her head because the light from her feet was emitted at an ealier time, $\tilde{t}_1$. Once she is inside the horizon, she sees her feet at a larger radius. Consider the moment when her head is at point A, the null geodesic connecting the world lines of her head and feet represents a photon emitted at B, which is a larger radius (emitted earlier in time). See figure 2 for an illustration, the world line of her head is shown in solid red, and the world line of her feet in dashed red.

3.b *Is it dark inside a black hole?* An observer outside sees a star collapsing to a black hole become dark. But would it be dark inside a black hole assuming a collapsing star continues to radiate at a steady rate as measured by an observer on its surface?

Using the same argument as above, the inside of a black hole would be light, as long as the observer started outside the star as it collapsed. Replace the world line of the observer's feet with the surface of the star in the above diagram. Once the observer has crossed the event horizon, there is still light coming from the surface of the star reaching her eyes.

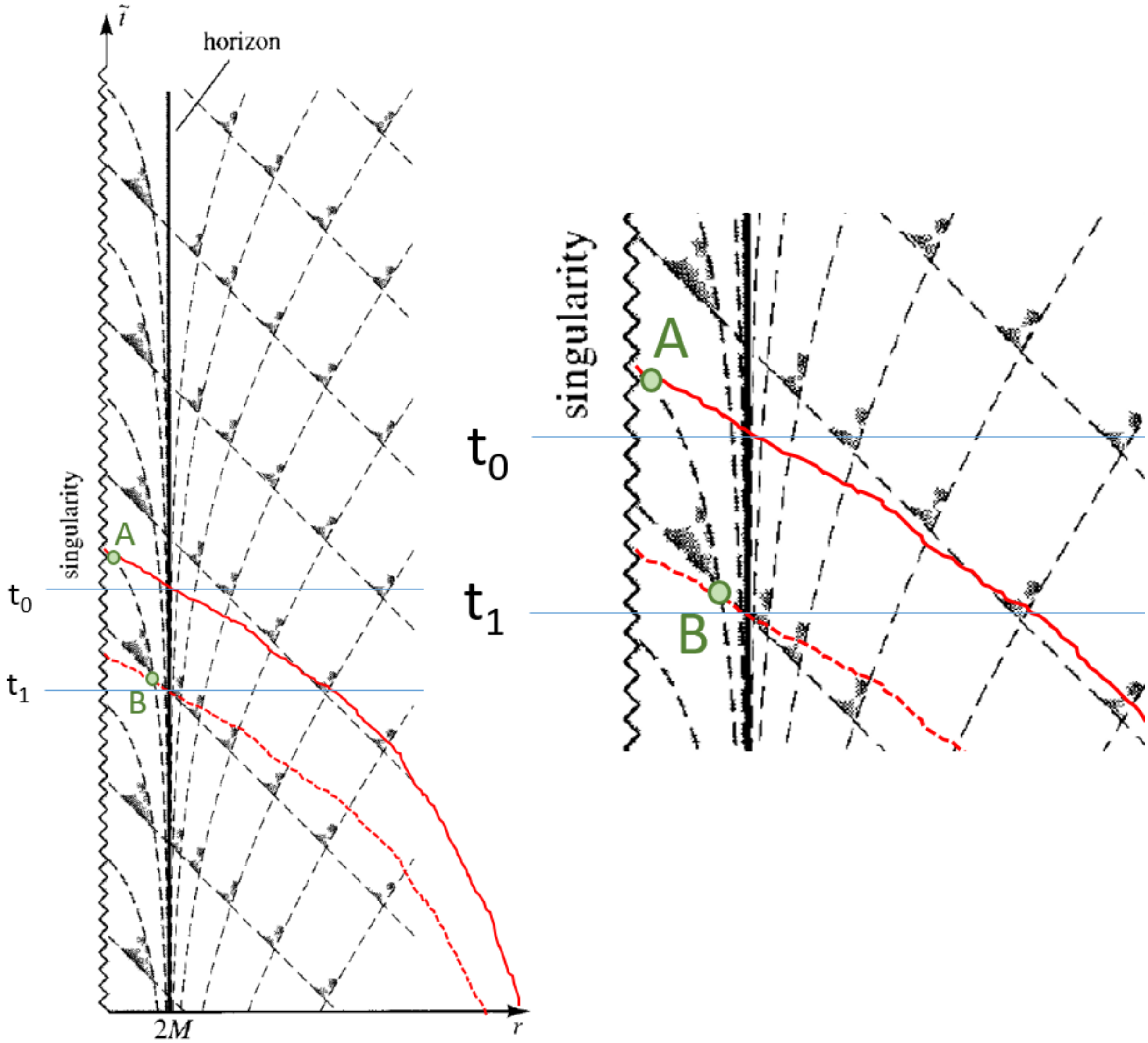

Figure 2: Eddington-Finkelstein diagram of an observer falling radially into a black hole feet-first. The dashed line indicates the world line of the observer's feet, and the solid line her head. At $\tilde{t}_0$, light emitted from the observer's feet at $\tilde{t}_1$ reaches her eyes, making it appear as if her feet are the same distance from the balck hole as her head. Light received at point A, emitted from point B causes it to appear that the observer's feet are at a larger radius than her head.

4 Hartle 12.15: Escaping a black hole.

A spaceship whose mission it is to study the environment around black holes is hovering at a Schwarzschild radius R outside a spherical black hole of mass M. To escape back to infinity, the crew must eject part of the rest mass of the ship to propel the remaining fraction to escape velocity. What is the largest fraction f of the rest mass that can escape to infinity? What happens to this fraction as R approaches $2M$?

Initially, the ship of mass m is at rest at radius R, so it has four velocity

$$u^\alpha_{\text{initial}} = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0 \right] , \tag{18}$$

coming from the normalization condition for timelike four-velocities. After ejecting some fraction of its mass, the ship has escaped in a purely radial trajectory (if the ejecta is sent radially inwards to the black hole), so the four-velocity is given by (Hartle 9.36), with the sign on u^r changed to reflect an outgoing trajectory. However, we also know the ship will be moving at escape velocity $v_{\text{esc}} = \sqrt{2M/R}$, so

$$u^\alpha_{\text{esc}} = \left[\left(1 - \frac{2M}{R}\right)^{-1}, \left(\frac{2M}{R}\right)^{1/2}, 0, 0 \right] , \tag{19}$$

but after ejecting some of its mass, the ship has mass fm. We can now conserve four-momentum:

$$m u^\alpha_{\text{initial}} = m f u^\alpha_{\text{esc}} + m_{\text{ej}} u^\alpha_{\text{ej}} , \tag{20}$$

where u^α_{ej} is the four-velocity of the ejecta, and m_{ej} is its rest mass. Note that we do not conserve rest mass. From the spatial components (conservation of three-momentum), we see

$$0 = f \left(\frac{2M}{R}\right)^{1/2} + \frac{m_{\text{ej}}}{m} u^r_{\text{ej}} \quad \Rightarrow \quad u^r_{\text{ej}} = -\frac{f}{m_{\text{ej}}/m} \left(\frac{2M}{R}\right)^{1/2} , \tag{21}$$

and that $u^\theta_{\text{ej}} = u^\phi_{\text{ej}} = 0$. Let us define $h \equiv m_{\text{ej}}/m$. Now for the temporal component (conservation of energy):

$$\left(1 - \frac{2M}{R}\right)^{-1/2} = f \left(1 - \frac{2M}{R}\right)^{-1} + h u^t_{\text{ej}} , \tag{22}$$

but we can determine the temporal component of the ejecta's four-velocity:

$$-1 = g_{\alpha\beta} u^\alpha_{\text{ej}} u^\beta_{\text{ej}} = -\left(1 - \frac{2M}{R}\right) \left(u^t_{\text{ej}}\right)^2 + \left(1 - \frac{2M}{R}\right)^{-1} \left(u^r_{\text{ej}}\right)^2 \tag{23}$$

$$= -\left(1 - \frac{2M}{R}\right) \left(u^t_{\text{ej}}\right)^2 + \left(1 - \frac{2M}{R}\right)^{-1} \left(\frac{f}{h}\right)^2 \left(\frac{2M}{R}\right) , \tag{24}$$

which yields

$$u^t_{\text{ej}} = \left(1 - \frac{2M}{R}\right)^{-1/2} \left\{ 1 + \left[1 - \frac{2M}{r}\right]^{-1} \left(\frac{f}{h}\right)^2 \left(\frac{2M}{R}\right) \right\}^{1/2} \tag{25}$$

$$= \left(1 - \frac{2M}{R}\right)^{-1} \left\{ 1 - \frac{2M}{R} \left[1 - \left(\frac{f}{h}\right)^2 \right] \right\}^{1/2} . \tag{26}$$

If we insert the above result into (22) and do some minor rearrangement, we find

$$\left(1-\frac{2M}{R}\right)^{1/2}=f+h\left\{1-\frac{2M}{R}\left[1-\left(\frac{f}{h}\right)^2\right]\right\}^{1/2} \quad . \tag{27}$$

Inserting for the masses explicitly:

$$\left(1-\frac{2M}{R}\right)^{1/2}=\left(\frac{m_{\mathrm{esc}}}{m}\right)+\left(\frac{m_{\mathrm{ej}}}{m}\right)\left\{1-\frac{2M}{R}\left[1-\left(\frac{m_{\mathrm{esc}}}{m_{\mathrm{ej}}}\right)^2\right]\right\}^{1/2} \quad . \tag{28}$$

Mathematica tells us

$$\begin{aligned} f &= \left(1-\frac{2M}{R}\right)^{-1}\left(\sqrt{1-\frac{2M}{R}}\pm\sqrt{\left(\frac{2M}{R}\right)^2 h^2-\left(\frac{2M}{R}\right)^2-2\left(\frac{2M}{R}\right)h^2+\left(\frac{2M}{R}\right)+h^2}\right) \\ &= \left(1-\frac{2M}{R}\right)^{-1}\left(\sqrt{1-\frac{2M}{R}}\pm\sqrt{\left[1-\frac{2M}{R}\right]\left[h^2\left(1-\frac{2M}{R}\right)+\frac{2M}{R}\right]}\right) \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\partial f}{\partial h} &= \pm h\left(1-\frac{2M}{R}\right)^{-1}\left[1-\frac{2M}{R}\right]^{-1/2}\left[h^2\left(1-\frac{2M}{R}\right)+\frac{2M}{R}\right]^{-1/2}\left[1-\frac{2M}{R}\right]^2 \\ &= \pm h\left[1-\frac{2M}{R}\right]^{+1/2}\left[h^2\left(1-\frac{2M}{R}\right)+\frac{2M}{R}\right]^{-1/2} \quad , \end{aligned} \tag{30}$$

and we see $\mathrm{d}f/\mathrm{d}h = 0$ when $h = 0$. This implies the largest fraction of the rest mass can escape when $h = 0$, or the ejecta is massless and the change in rest mass is converted to energy. Inserting $h = 0$ into our expression for f yields

$$f=\left(1-\frac{2M}{R}\right)^{1/2}\left(1+\left[\frac{2M}{R}\right]^{1/2}\right)^{-1} \quad , \tag{31}$$

which vanishes for $R = 2M$.

5 Hartle 14.6: Geodetic precession of circular orbit.

What is the largest possible geodetic precession for a stable circular orbit in the Schwarzschild geometry?

The geodetic precession of a gyroscope on a circular orbit is given by (Hartle 14.18):

$$\Delta\phi_{\text{geodetic}} = 2\pi \left[1 - \left(1 - \frac{3M}{r}\right)^{1/2}\right] \ , \tag{32}$$

which we see is proportional to $r^{-1/2}$. Thus, smaller orbits lead to larger geodetic precession. The smallest stable circular orbit in the Schwarzschild geometry occurs at $r = r_{\text{ISCO}} = 6M$, for fixed M. Inserting the radius of the orbit yields

$$\Delta\phi_{\text{geodetic}} = 2\pi \left[1 - \frac{1}{\sqrt{2}}\right] \simeq 1.84\pi = 105.4^\circ \ , \tag{33}$$

per orbit.

6 Hartle 14.9: Rotational effects of the Sun.

General relativity predicts that, because the sun is rotating, a light ray passing by will be deflected slightly by an amount additional to the deflection of light in the Schwarzschild geometry considered in Hartle Section 9.4. Calculate the amount and direction of this deflection to lowest nonvanishing order in $1/c$ assuming that the orbit is in the equatorial plane perpendicular to the axis of rotation. Estimate the magnitude of this effect for the Sun. Is it an important correction to the results of the observations discussed in Hartle Section 10.3? (*Hint*: Before doing any algebra, think about the what terms in the metric wil contribute to the final answer in leading order in $1/c$.)

The sun is rotating slowly, so the metric of the spacetime to its exterior is given by (Hartle 14.22):

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\text{sch}} - \frac{4GJ}{c^3 r^2}\sin^2\theta (r\mathrm{d}\phi)(c\mathrm{d}t) + \mathcal{O}(J^2) \ , \tag{34}$$

where J is the angular momnentum of the rotating body, and

$$\mathrm{d}s^2_{\text{sch}} = -\left(1 - \frac{2GM}{rc^2}\right)(c\mathrm{d}t)^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \ , \tag{35}$$

is the spherically symmetric Schwarzschild geometry. We can ignore the second terms in the coefficients of $\mathrm{d}t^2$ and $\mathrm{d}r^2$ in the Schwarzschild metric because the answer must be proprtional to J. Thus these terms would enter as multiplicative factors and their contribution would be proportional to at least $1/c^5$.

Let us select an equitorial orbit ($\theta = \pi/2$), such that

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 - \frac{4J}{r}\mathrm{d}\phi\mathrm{d}t + \mathcal{O}(J^2) \ , \tag{36}$$

where we have removed the factors of G and c, by setting them to unity. Thus we can write the metric as

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & -2\frac{J}{r} \\ 0 & 1 & 0 \\ -2\frac{J}{r} & 0 & r^2 \end{pmatrix} \ , \tag{37}$$

where we have dropped the θ coordinate. Just as in the non-rotating Schwarzschild metric, this is cyclic in ϕ and t, so we can define the following Killing vectors and their associated conserved quantities:

$$\boldsymbol{\xi} = (1,0,0,0) \quad \Rightarrow \quad \varepsilon = -\boldsymbol{\xi}\cdot\mathbf{u} = -g_{\alpha t}u^\alpha = u^t + 2\frac{J}{r}u^\phi \tag{38}$$

$$\boldsymbol{\eta} = (0,0,0,1) \quad \Rightarrow \quad \ell = \boldsymbol{\eta}\cdot\mathbf{u} = g_{\alpha\phi}u^\alpha = r^2u^\phi - 2\frac{J}{r}u^t \ , \tag{39}$$

where $\mathbf{u} = \mathrm{d}\mathbf{x}/\mathrm{d}\lambda$ is the four-velocity of the particle in motion, in this case, a photon (λ is an affine parameter). Using the normalization condition for photons $\mathbf{u}\cdot\mathbf{u} = 0$, we find the radial equation:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 + \frac{4J}{r}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right) \ . \tag{40}$$

We can solve (38) and (39) for u^t and u^ϕ and insert into the above to find an equation only in r. Solving the equations yields

$$u^t = \frac{r(r^3\varepsilon - 2J\ell)}{4J^2 + r^4} \simeq \varepsilon - \frac{2J\ell}{r^3} \tag{41}$$

$$u^\phi = \frac{r(2J\varepsilon + \ell r)}{4J^2 + r^4} \simeq \frac{2J\varepsilon}{r^3} + \frac{\ell}{r^2} \ , \tag{42}$$

after dropping terms quadratic in J. Inserting these into the radial equation, we find

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \left(\varepsilon - \frac{2J\ell}{r^3}\right)^2 - r^2\left(\frac{2J\varepsilon}{r^3} + \frac{\ell}{r^2}\right)^2 + \frac{4J}{r}\left(\varepsilon - \frac{2J\ell}{r^3}\right)\left(\frac{2J\varepsilon}{r^3} + \frac{\ell}{r^2}\right) \tag{43}$$

$$= \epsilon^2 - \frac{1}{r^2}\left(\frac{4J}{r}\ell\varepsilon + \ell^2\right) = \epsilon^2 - \frac{\ell^2}{r^2}\left(\frac{4J}{r}\frac{\varepsilon}{\ell} + 1\right) \ . \tag{44}$$

We can rearrange the above to be

$$\frac{\varepsilon^2}{\ell^2} = \frac{1}{\ell^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{1}{r^2}\left(\frac{4J}{r}\frac{\varepsilon}{\ell} + 1\right) \ , \tag{45}$$

and we can define the impact parameter $b = \ell/\varepsilon$, and identify the second term on the right-hand side as an effective potential $W_{\text{eff}}(r)$:

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \ell\left\{\frac{1}{b^2} - W_{\text{eff}}(r)\right\}^{1/2} \quad \text{with} \quad W_{\text{eff}}(r) = \frac{1}{r^2}\left(\frac{4J}{r}\frac{1}{b} + 1\right) \ . \tag{46}$$

Following the treatment in Hartle section 9.4 (The Deflection of Light, page 210), we want to find the shape of the orbit:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\mathrm{d}\phi/\mathrm{d}\lambda}{\mathrm{d}r/\mathrm{d}\lambda} = \left(\frac{2J\varepsilon}{r^3} + \frac{\ell}{r^2}\right)\left\{\varepsilon^2 - \frac{\ell^2}{r^2}\left(\frac{4J}{r}\frac{\varepsilon}{\ell} + 1\right)\right\}^{-1/2} \ , \tag{47}$$

or equivalently

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{1}{r^2}\left(\frac{2J}{r}\frac{1}{b} + 1\right)\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(\frac{4J}{r}\frac{1}{b} + 1\right)\right\}^{-1/2} \ . \tag{48}$$

Thus we find the azimuthal angle of the photon's trajectory as a function of its Schwarzschild radial position:

$$\phi = \int_{-\infty}^{\infty} \frac{\mathrm{d}r}{r^2}\left(\frac{2J}{br} + 1\right)\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(\frac{4J}{br} + 1\right)\right\}^{-1/2} \ . \tag{49}$$

As in Hartle, let us define the radial coordinate of the point of closest approach as r_1, then due to the symmetry of the problem, the integral becomes

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{\mathrm{d}r}{r^2}\left(\frac{2J}{br} + 1\right)\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(\frac{4J}{br} + 1\right)\right\}^{-1/2} \ . \tag{50}$$

Above, r_1 is the turning point, given by (45) at $r = r_1$, where $\mathrm{d}r/\mathrm{d}\lambda = 0$: $W_{\text{eff}}(r_1) = 1/b^2$. Still following Hartle, we define a new integration variable $w = b/r$, making the integral

$$\Delta\phi = 2\int_{w_1}^{0}\left(-\frac{\mathrm{d}w}{b}\right)\left(\frac{2J}{b^2}w+1\right)\left\{\frac{1}{b^2}-\frac{w^2}{b^2}\left(\frac{4J}{b^2}w+1\right)\right\}^{-1/2} \tag{51}$$

$$= 2\int_{0}^{w_1}\mathrm{d}w\left(\frac{2J}{b^2}w+1\right)\left\{1-\left(\frac{4J}{b^2}w+1\right)w^2\right\}^{-1/2} . \tag{52}$$

We can rewrite this as

$$\Delta\phi = 2\int_0^{w_1}\mathrm{d}w\left(1+\frac{2J}{b^2}w\right)\left(1+\frac{4J}{b^2}w\right)^{-1/2}\left\{\left(1+\frac{4J}{b^2}w\right)^{-1}-w^2\right\}^{-1/2} . \tag{53}$$

Let us take $J/b^2 \ll 1$, such that we can express the integrand $\mathcal{I}$ as

$$\mathcal{I} = \left(1+\frac{2J}{b^2}w\right)\left(1-\frac{2J}{b^2}w\right)\left\{\left(1-\frac{4J}{b^2}w\right)-w^2\right\}^{-1/2} \tag{54}$$

$$= \left(1-\left[\frac{2J}{b^2}w\right]^2\right)\left\{\left(1-\frac{4J}{b^2}w\right)-w^2\right\}^{-1/2} , \tag{55}$$

which allows us to write the integral as

$$\Delta\phi = 2\int_0^{w_1}\mathrm{d}w\frac{1-(2J/b^2)^2w^2}{[1-(4J/b^2)w-w^2]^{1/2}} \tag{56}$$

$$= 2\int_0^{w_1}\frac{\mathrm{d}w}{[1-(4J/b^2)w-w^2]^{1/2}} - 2\left(\frac{2J}{b^2}\right)^2\int_0^{w_1}\frac{w^2\mathrm{d}w}{[1-(4J/b^2)w-w^2]^{1/2}} . \tag{57}$$

To leading order in J, we can neglect the second term in the above integral. The turning point w_1 is given by the radial equation using the turning point condition $\mathrm{d}r/\mathrm{d}\lambda = 0$:

$$\frac{1}{b^2} = W_{\text{eff}}(w_1) \quad \Rightarrow \quad 1 = w_1^2\left(\frac{4J}{b^2}w_1+1\right) , \tag{58}$$

alternatively, w_1 is also a root of the denominator in the integral, which is easier to solve, being one order less in w_1. The good ol' quadratic formula gives us two solutions, only one of which is positive:

$$w_1 = \frac{\sqrt{b^4+4J^2}}{b^2} - \frac{2J}{b^2} \simeq 1 - \frac{2J}{b^2} , \tag{59}$$

after dropping the J^2 term. Let us define the following:

$$X = 1-(4J/b^2)w-w^2 \quad \text{and} \quad q = -4-(4J/b^2)^2 = -4\left(1-4\frac{J^2}{b^4}\right) \simeq -4 , \tag{60}$$

in the limit $J \ll b^2$. Then we can write the previous as

$$\Delta\phi = 2\int_0^{w_1}\frac{\mathrm{d}w}{\sqrt{X}} = -2\frac{1}{\sqrt{-(-1)}}\arcsin\left(\frac{2(-1)x+\left(-\frac{4J}{b^2}\right)}{\sqrt{-q}}\right) = -2\arcsin\left(-\left[w+\frac{2J}{b^2}\right]\right) , \tag{61}$$

evaluated from 0 rto w_1. We can expand the arcsine noting $\arcsin(x) = x + \mathcal{O}(x^3)$, so:

$$\Delta\phi = 2\left[w + \frac{2J}{b^2}\right]_0^{w_1} = 2w_1 = 2\left(1 - \frac{2J}{b^2}\right) \ . \tag{62}$$

Treating our 1 as π, we find the change in deflection angle is

$$\delta\phi_{\rm rot} = -\frac{4J}{b^2} = -\frac{J}{Mb}\left(\frac{4M}{b}\right) = -\left(\frac{J}{Mb}\right)\delta\phi_{\rm def} \ , \tag{63}$$

the quantity of J/Mb is unitless, so it is a velocity, namely the velocity of the surface of the star. For the Sun the surface speed is ~ 2 km/s, so this effect is indeed very small.

7 Hartle 15.5: Null trajectories in the Kerr geometry.

The null directions on the horizon of a rotating black hole were identified in (Hartle 15.10). But does a light ray that starts out on one of these directions remain on the horizon? Use the geodesic equation for light rays in the Kerr geometry to show that it does. Show also that the light ray remains at a fixed value of θ.

A null trajectory in the Kerr geometry satisfies the geodesic equation for light:

$$\frac{\mathrm{d}\ell^\delta}{\mathrm{d}\lambda} = -\Gamma^\delta{}_{\beta\gamma}\ell^\beta\ell^\gamma \quad \text{with} \quad \ell^\alpha = (1, 0, 0, \Omega_H) \;, \tag{64}$$

where λ is an affine parameter along the photon's trajectory and with Ω_H given by (83). Using the definition of the Christoffel symbols, the above can be written as

$$\frac{\mathrm{d}\ell^\delta}{\mathrm{d}\lambda} = -\frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\ell^\beta\ell^\gamma = -\frac{1}{2}g^{\alpha\delta}\left(2\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\ell^\beta\ell^\gamma \;, \tag{65}$$

due to the symmetry of $g_{\alpha\beta}$ and $\ell^\alpha\ell^\beta$ under $\alpha \leftrightarrow \beta$. A null trajectory starting with Boyer-Lindquist θ will remain at θ iff

$$\begin{aligned}\frac{\mathrm{d}\ell^\theta}{\mathrm{d}\lambda} = 0 = -\frac{1}{2}g^{\alpha\theta}\left(2\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\ell^\beta\ell^\gamma &= -\frac{1}{2}g^{\theta\theta}\left(2\frac{\partial g_{\theta\beta}}{\partial x^\gamma}\ell^\beta\ell^\gamma - \frac{\partial g_{\gamma\beta}}{\partial x^\theta}\ell^\beta\ell^\gamma\right) \\ &= -\frac{1}{2}g^{\theta\theta}\left(2\frac{\partial g_{\theta\theta}}{\partial x^\gamma}\ell^\theta\ell^\gamma - \frac{\partial g_{\gamma\beta}}{\partial x^\theta}\ell^\beta\ell^\gamma\right) \;,\end{aligned} \tag{66}$$

due to there being no off-diagonal θ components to the metric. The first term in the last line above vanishes becuase $\ell^\theta = 0$. Carrying out the sum implied by the last term we see

$$\frac{\mathrm{d}\ell^\theta}{\mathrm{d}\lambda} = \frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{\gamma\beta}}{\partial x^\theta}\ell^\beta\ell^\gamma\right) = \frac{1}{2}g^{\theta\theta}\left(\frac{\partial g_{tt}}{\partial x^\theta}\ell^t\ell^t + \frac{\partial g_{\phi\phi}}{\partial x^\theta}\ell^\phi\ell^\phi + 2\frac{\partial g_{t\phi}}{\partial x^\theta}\ell^t\ell^\phi\right) \tag{67}$$

$$= \frac{1}{2}g^{\theta\theta}\frac{\partial}{\partial\theta}\left(g_{tt}\ell^t\ell^t + g_{\phi\phi}\ell^\phi\ell^\phi + 2g_{t\phi}\ell^t\ell^\phi\right) = \frac{1}{2}g^{\theta\theta}\frac{\partial}{\partial\theta}\left(\boldsymbol{\ell}\cdot\boldsymbol{\ell}\right) \;, \tag{68}$$

and since $\boldsymbol{\ell}$ is a null vector we see

$$\frac{\mathrm{d}\ell^\theta}{\mathrm{d}\lambda} = \frac{1}{2}g^{\theta\theta}\frac{\partial}{\partial\theta}\left(\boldsymbol{\ell}\cdot\boldsymbol{\ell}\right) = \frac{1}{2}g^{\theta\theta}\frac{\partial}{\partial\theta}(0) = 0 \;. \tag{69}$$

The Kerr horizon is a null surface of constant r, so for a trajectory on the surface to remain on the surface, we enforce

$$0 = \left.\frac{\mathrm{d}\ell^r}{\mathrm{d}\lambda}\right|_{r=r_+} = -\frac{1}{2}g^{\alpha r}\left(2\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\ell^\beta\ell^\gamma\bigg|_{r=r_+} = -\frac{1}{2}g^{rr}\left(2\frac{\partial g_{r\beta}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^r}\right)\ell^\beta\ell^\gamma\bigg|_{r=r_+} \tag{70}$$

$$= -\frac{1}{2}g^{rr}\left(2\frac{\partial g_{rr}}{\partial x^\gamma}\ell^r\ell^\gamma - \frac{\partial g_{\gamma\beta}}{\partial x^r}\ell^\beta\ell^\gamma\right)\bigg|_{r=r_+} = \frac{1}{2}g^{rr}\left(\frac{\partial g_{\gamma\beta}}{\partial x^r}\ell^\beta\ell^\gamma\right)\bigg|_{r=r_+} \;, \tag{71}$$

the first term vanishes again because $\ell^r = 0$. Let us investigate the quantity

$$g^{rr}\bigg|_{r=r_+} = \frac{\Delta}{\rho^2}\bigg|_{r=r_+} = \frac{r_+^2 - 2Mr_+ + a^2}{r_+^2 + a^2\cos^2\theta} \;, \tag{72}$$

using the fact that $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_{\ \gamma}$ which is unity if $\alpha = \gamma$ and zero otherwise (Kronecker delta). Note the numerator vanishes by (85), so we have shown

$$\frac{\mathrm{d}\ell^r}{\mathrm{d}\lambda} = 0 \;. \tag{73}$$

8 Hartle 15.6: Orthogonality in Kerr space.

Show explicitly that the two vectors (0,0,1,0) and (0,0,0,1) on the horizon $r = r_+$ are (a) spacelike and (b) orthogonal to each other and to the null generator ℓ (Hartle 15.10).

The Kerr metric is given by (Hartle 15.1) and (Hartle 15.2), and will not be reproduced here in full, only appearing when specific terms are required for a calculation. Let us define the four-vectors

$$\mathbf{A} = (0,0,1,0) \quad \text{and} \quad \mathbf{B} = (0,0,0,1) \ . \tag{74}$$

For these to be spacelike, we must show

$$\mathbf{A}\cdot\mathbf{A} = g_{\alpha\beta}A^\alpha A^\beta > 0 \quad \text{and} \quad g_{\alpha\beta}B^\alpha B^\beta > 0 \ . \tag{75}$$

Let us work on the vector A first. Here, the only non-zero components is the third, so

$$\mathbf{A}\cdot\mathbf{A} = g_{\alpha\beta}A^\alpha A^\beta = g_{\theta\theta}\left(A^\theta\right)^2 = g_{\theta\theta} = \rho^2 \ , \tag{76}$$

where J is the anglar momentum and M is the mass of the black hole. For the vector B, we have

$$\mathbf{B}\cdot\mathbf{B} = g_{\alpha\beta}B^\alpha B^\beta = g_{\phi\phi}\left(A^\phi\right)^2 = g_{\phi\phi} = \sin^2\theta\left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right) \ . \tag{77}$$

In the above, we have made the definitions:

$$a \equiv J/M \qquad \rho^2 \equiv r^2 + a^2\cos^2\theta \ . \tag{78}$$

The Boyer-Lindquist radius of the horizon of a Kerr black hole is given by (Hartle 15.6):

$$r_+ = M + \sqrt{M^2 - a^2} \ . \tag{79}$$

Thus combining our definitions and evaluating at $r = r_+$ which is positive definite, we find

$$\mathbf{A}\cdot\mathbf{A} = r_+^2 + a^2\cos^2\theta > 0 \tag{80}$$

$$\mathbf{B}\cdot\mathbf{B} = \sin^2\theta\left(r_+^2 + a^2 + (Mr_+)\frac{2a^2\sin^2\theta}{\rho^2}\right) > 0 \ . \tag{81}$$

To show **A** and **B** are orthogonal to each other, we simply calculate their scalar product and show it is zero:

$$\mathbf{A}\cdot\mathbf{B} = g_{\alpha\beta}A^\alpha B^\beta = g_{\theta\beta}B^\beta = g_{\theta\phi} = 0 \ , \tag{82}$$

since there are no terms proportional to $\mathrm{d}\theta\mathrm{d}\phi$ in the Kerr line element (Hartle 15.1), and we obtain our desired result. The null tangent vector $\boldsymbol{\ell}$, given by (Hartle 15.10), is

$$\ell = (1,0,0,\Omega_H) \quad \text{with} \quad \Omega_H = \frac{a}{2Mr_+} \ . \tag{83}$$

It is trivial to show **A** is orthogonal to $\boldsymbol{\ell}$ since A^θ is the only non-zero component and $\ell^\theta = 0$. The final scalar product of interest is:

$$\boldsymbol{\ell}\cdot\mathbf{B} = g_{\alpha\beta}\ell^\alpha B^\beta = g_{\phi\beta}\ell^\beta = g_{\phi t}\ell^t + g_{\phi\phi}\ell^\phi = g_{\phi t} + \Omega_H g_{\phi\phi} \ . \tag{84}$$

Let us note that $r_+^2 = 2M(M + \sqrt{M^2 - a^2}) - a^2$, so

$$r_+^2 + a^2 - 2Mr_+ = 0 \ . \tag{85}$$

and we that we can write

$$\rho^2 = r^2 + a^2 + a^2(\cos^2\theta - 1) = r^2 + a^2 - a^2\sin^2\theta \quad \Rightarrow \quad \rho^2|_{r=r_+} = 2Mr_+ - a^2\sin^2\theta \ . \tag{86}$$

The relevant components of the metric, evaluated at the Kerr horizon are

$$g_{\phi t} = -\left(\frac{1}{2}\right)\frac{4Mar_+\sin^2\theta}{\rho^2|_{r=r+}} \tag{87}$$

$$g_{\phi\phi} = \sin^2\theta\left(r_+^2 + a^2 + \frac{2Mr_+a^2\sin^2\theta}{\rho^2|_{r=r_+}}\right) = \frac{2Mr_+\sin^2\theta}{\rho^2|_{r=r_+}}\left(\rho^2|_{r=r_+} + a^2\sin^2\theta\right) \ , \tag{88}$$

where the factor of 1/2 comes from the symmetry of $g_{\alpha\beta}$ and we've done the last step by pulling out a common denominator and using (85) to eliminate $r_+^2 + a^2$. Thus

$$\boldsymbol{\ell}\cdot\mathbf{B} = g_{\phi t} + \Omega_H g_{\phi\phi} = \frac{a\sin^2\theta}{\rho^2|_{r=r+}}\left\{-2Mr_+ + \left(\rho^2|_{r=r_+} + a^2\sin^2\theta\right)\right\} \tag{89}$$

$$= \frac{a\sin^2\theta}{\rho^2|_{r=r+}}\left\{-2Mr_+ + \left(2Mr_+ - a^2\sin^2\theta\right) + a^2\sin^2\theta\right\} = 0 \ , \tag{90}$$

as expected.

9 Hartle 15:11: Circular photon orbits in Kerr space.

Show that in the geometry of an extremal Kerr black hole of mass M there are circular light ray orbits in the equatorial plane at Boyer-Lindquist radii $r = M$ rotating with the black hole (corotating) and $r = 4M$ in the opposite direction (counterrotating).

In the equitorial plane $\theta = \pi/2$, so that $\rho^2 = r^2$, and the Kerr line element becomes

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \frac{4Ma}{r}\mathrm{d}\phi\mathrm{d}t + \frac{r^2}{\Delta}\mathrm{d}r^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\mathrm{d}\phi^2 \ , \tag{91}$$

and is given in (Hartle 15.14). This leads to an effective potential for photons given by (Hartle 15.22):

$$W_{\text{eff}}(r,\sigma,b) = \frac{1}{r^2}\left[1 - \left(\frac{a}{b}\right)^2 - \frac{2M}{r}\left(1 - \sigma\frac{a}{b}\right)^2\right] \ , \tag{92}$$

where $\sigma \equiv \text{sign}(\ell)$ and $b \equiv |\ell/e|$ with the conserved quantities ℓ, e as previously defined by the associated Killing vectors as in other assignments. The condition for circular orbits at radius r_0 comes from (Hartle 15.21) with $\mathrm{d}r/\mathrm{d}\lambda = 0$:

$$\frac{1}{b^2} = W_{\text{eff}}(r_0, \sigma, b) \ , \tag{93}$$

as well as the fact that we must sit at the extrema of the potential:

$$0 = \left.\frac{\partial W_{\text{eff}}}{\partial r}\right|_{r=r_0} = \frac{2}{b^2 r_0^4}\left(3M(\sigma a - b)^2 + (a^2 - b^2)r_0\right) \ . \tag{94}$$

We are interested in an extremal Kerr black hole, so $a = M$, and our conditions on the orbits become

$$0 = \frac{2}{b^2 r_0^4}\left(3M(\sigma M - b)^2 + (M^2 - b^2)r_0\right) \tag{95}$$

$$\frac{1}{b^2} = \frac{1}{r_0^2}\left[1 - \left(\frac{M}{b}\right)^2 - \frac{2M}{r_0}\left(1 - \sigma\frac{M}{b}\right)^2\right] \ . \tag{96}$$

Beating MATHEMATICA into combining the above equations to eliminate b yields

$$4M^3 = r_0^3 - 6Mr_0^2 + 9M^2 r_0 \ , \tag{97}$$

for both $\sigma = \pm 1$, which has roots $r_0 = M,\ 4M$.

10 Hartle 15.12: Angular velocity in Kerr space.

The angular velocity $\Omega = \mathrm{d}\phi/\mathrm{d}t$ of circular orbits of Boyer-Lindquist radius r in the Kerr geometry is given by the simple formula

$$\Omega = \pm \frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}} \ . \tag{98}$$

Here the upper sign refers to corotating orbits and the lower one to counterrotating orbits. Explain how to derive this formula, and exhibit the algebraic equations from which it follows. However, don't try to solve the equations unless you really like algebra!

Starting from the line element of the Kerr geometry, we can identify the Lagrangian as well as the cyclic coordinates t and ϕ. This leads us to the two Killing vectors and resultant conserved quantities ℓ and e:

$$-e = \boldsymbol{\xi} \cdot \mathbf{u} = g_{tt}u^t + g_{t\phi}u^\phi \tag{99}$$

$$\ell = \boldsymbol{\eta} \cdot \mathbf{u} = g_{\phi t}u^t + g_{\phi\phi}u^\phi \ , \tag{100}$$

In the end we are looking for the angular velocity:

$$\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}\phi/\mathrm{d}\lambda}{\mathrm{d}t/\mathrm{d}\lambda} = \frac{u^\phi}{u^t} \ , \tag{101}$$

where λ is an affine parameter along a null trajectory or the proper time along a timelike trajectory. Let us factor out u^t from the conserved quantities:

$$\frac{-e}{u^t} = g_{tt} + g_{t\phi}\frac{u^\phi}{u^t} \tag{102}$$

$$\frac{\ell}{u^t} = g_{\phi t} + g_{\phi\phi}\frac{u^\phi}{u^t} \ , \tag{103}$$

then take their ratio:

$$-\frac{e}{\ell} = \frac{g_{tt} + g_{t\phi}\frac{u^\phi}{u^t}}{g_{\phi t} + g_{\phi\phi}\frac{u^\phi}{u^t}} = \frac{g_{tt} + \Omega g_{t\phi}}{g_{\phi t} + \Omega g_{\phi\phi}} \ , \tag{104}$$

and then the algebra begins. The ideal path would be to find e and ℓ in terms of black hole parameters M and a. Then the above can be solved for Ω in terms of M, a, and the radius of the circular orbit r.