DYLAN J. TEMPLES: SOLUTION SET SIX

Northwestern University PHYS 445, General Relativity
Gravity: An Introduction to Einstein's General Relativity - Hartle
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Contents

1 Hartle 18.3: Comoving frames in the FRW cosmological model.

Consider a flat FRW model whose metric is given by (Hartle 18.1). Show that, if a particle is shot from the origin at time t_* with a speed V_* as measured by a comoving observer (constant x, y, z), then asymptotically it comes to rest with respect to a comoving frame. Express the comoving coordinate radius at which it comes to rest as an integral over $a(t)$.

Let us align the z-axis of our reference frame with the particle's motion, thus the FRW line element becomes two-dimensional:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}z^2 \ , \tag{1}$$

which is independent of z. Thus the killing vector $\boldsymbol{\xi} = (0, 1)$ implies momentum in the z direction is conserved:

$$\boldsymbol{\xi} \cdot \mathbf{u} = a^2(t)\frac{\mathrm{d}z}{\mathrm{d}\tau} \equiv p \ , \tag{2}$$

if the particle's four-momentum is $\mathbf{u}$. The normalization of the particle's four-velocity tells us

$$-1 = \mathbf{u} \cdot \mathbf{u} = -\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + a^2(t)\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \ . \tag{3}$$

From each of the above, it is clear

$$u_z = \frac{p}{a^2(t)} \tag{4}$$

$$u_t = \sqrt{\frac{p^2}{a^2(t)} + 1} \ . \tag{5}$$

Now that we have the components of the particle's four-velocity, we can find the particle's three velocity:

$$v_z = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{u_z}{u_t} = \frac{p/a^2(t)}{\sqrt{1 + p^2/a^2(t)}} \ . \tag{6}$$

After a time T, the particle has reached a radius of

$$z = \int_{t_*}^{T} \mathrm{d}t\frac{\mathrm{d}z}{\mathrm{d}t} = \int_{t_*}^{T} \mathrm{d}t\left(\frac{p}{a^2(t)}\right)\left[1 + \left(\frac{p}{a^2(t)}\right)\right]^{-1/2} \tag{7}$$

if, in these coordinates, the particle eventually comes to rest the above is equivalent to

$$z = \int_{t_*}^{\infty} \mathrm{d}t\left(\frac{p}{a^2(t)}\right)\left[1 + \left(\frac{p}{a^2(t)}\right)\right]^{-1/2} \tag{8}$$

A comoving observer has orthonormal basis vectors given by

$$\mathbf{e}_{\hat{t}} = (1, 0) \tag{9}$$

$$\mathbf{e}_{\hat{z}} = \frac{1}{a(t)}(0, 1) \ , \tag{10}$$

using the normalization condition $\mathbf{e}\cdot\mathbf{e} = -1$. The initial four-velocity along the $\hat{z}$ axis as measured by an observer with these basis vectors is

$$u_{*\hat{z}} = \mathbf{u}_* \cdot \mathbf{e}_{\hat{z}} = a^2(t_*)u_z \frac{1}{a(t_*)} = \frac{p}{a(t_*)} \ . \tag{11}$$

Locally the four velocity can be expressed as in Minkowski space:

$$\mathbf{u}_* = \frac{1}{\sqrt{1-V_*^2}}(1, \vec{V}_*) \ , \tag{12}$$

where $\vec{V}_* = V_*\hat{z}$. Thus the final radius where the particle comes to rest can be expressed

$$z = \int_{t_*}^{\infty} \left(\frac{\mathrm{d}t}{a(t)}\right) \frac{V_*}{\sqrt{1-V_*^2}} \left[1 + \left(\frac{1}{a(t)}\right) \frac{V_*}{\sqrt{1-V_*^2}}\right]^{-1/2} \ . \tag{13}$$

2 Hartle 18.7: Proper distance.

Consider a galaxy whose light we see today at time t_0 that was emitted at time t_e. Show that the present proper distance to the galaxy (along a curve of constant t_0) is

$$d = a(t_0) \int_{t_e}^{t_0} \mathrm{d}t/a(t) \ . \tag{14}$$

Let us use (t, χ, θ, ϕ) coordinates such that the line element is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \left[\mathrm{d}\chi^2 + \chi^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \right] \ , \tag{15}$$

thus a radially moving photon obeys

$$0 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\chi^2 \ . \tag{16}$$

A point labeled with comoving coordinate χ_e will always be labeled as such. A photon emitted at χ_e at time t_e and received at $\chi = 0$ and time t_0 will have traveled a distance

$$\chi_e = \int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} \ , \tag{17}$$

from rearranging and integrating the line photon trajectory (16), in agreement with (Hartle 18.7). The three-dimensional distance at time t_0 is then given by

$$d = \int_0^{\chi_e} \sqrt{g_{ij}} \mathrm{d}x_i \mathrm{d}x_j \bigg|_{t=t_0} = \int_0^{\chi_e} \sqrt{g_{zz}} \mathrm{d}z \bigg|_{t=t_0} = \int_0^{\chi_e} a(t_0) \mathrm{d}z = a(t_0)\chi_e \ , \tag{18}$$

and inserting for χ_e, we have

$$d = a(t_0) \int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} \ . \tag{19}$$

3 Hartle 18.8: Cosmological redshift.

In Hartle Section 9.2 the redshift of a photon in the Schwarzschild geometry was derived using the conservation law arising from time-translation symmetry. Show that the cosmological red shift (Hartle 18.10) can be derived from the *space* translation symmetry of the metric (Hartle 18.1) in a similar way.

Let us work with a FRW geometry with a flat spatial component in Cartesian coordinates, where the z axis is aligned with a photon's three-momentum. In this case the Killing vector $\boldsymbol{\xi} = (0, 0, 0, 1)$ leads to a conserved quantity:

$$q \equiv \mathbf{p} \cdot \boldsymbol{\xi} = g_{zz} p_z \xi_z = a^2(t) p_z(t) \ , \tag{20}$$

where $\mathbf{p}$ is the photon's four-momentum. From this we can see that $p_z(t)$ varies inversely with $a^2(t)$ over time. Additionally, the photon's four-momentum is null, so

$$0 = \mathbf{p} \cdot \mathbf{p} = g_{tt} p_t^2 + g_{zz} p_z^2 = -p_t^2 + a^2(t) p_z^2 = -p_t^2 + a^2(t) \frac{q^2}{a^4(t)} = -p_t^2 + \frac{q^2}{a^2(t)} \ , \tag{21}$$

so p_t varies inversely with $a(t)$ over time. A stationary observer has four velocity $\mathbf{u}_{\text{obs}} = (1, 0, 0, 0)$ and thus observes the photon to have energy

$$E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}} = -(-1) p_t = \frac{q}{a(t)} = \hbar \omega(t) \quad \Rightarrow \quad \omega(t) = \frac{q}{\hbar} \frac{1}{a(t)} \ , \tag{22}$$

and thus cosmological redshift is given by the relation

$$\frac{\omega_1}{\omega_2} = \frac{a(t_2)}{a(t_1)} \ , \tag{23}$$

where $\omega_{1(2)}$ is the frequency at time $t_{1(2)}$.

4 Hartle 18.15: Homogeneous and isotropic cosmological model.

Consider a homogeneous, isotropic, cosmological model described by the line element

$$ds^2 = -dt^2 + \left(\frac{t}{t_*}\right)\left[dx^2 + dy^2 + dz^2\right] , \tag{24}$$

where t_* is a constant.

4.a Is this model open, closed, or flat?

The unified form for the line element in FRW metrics of the three possibilities for curvature is given by (Hartle 18.62):

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] , \tag{25}$$

where the curvature is determined by k. A flat universe has $k = 0$, and $k = \pm 1$ corresponds to closed(+) and open(-) respectively. It is clear that for $k = 0$, the quantity inside the square brackets reduces to spherical polar coordinates, which is equivalent to Cartesian coordinates as in the given metric.

4.b Is this a matter-dominated universe? Explain.

From the metric it is clear

$$a(t) = \sqrt{t/t_*} , \tag{26}$$

thus $a(t) \propto t^{1/2}$. According to (Hartle 18.37), a matter-dominated universe requires $a(t) \propto (t/t_0)^{2/3}$, which the above is not. Furthermore, from (Hartle 18.38) we can conclude this universe is radiation-dominated.

4.c Assuming the Friedman equation holds for the universe, find $\rho(t)$.

The Friedman Equation is given by (Hartle 18.63), which for a flat universe becomes

$$\dot{a} = \sqrt{\frac{8\pi\rho}{3}}a . \tag{27}$$

This can be rearranged to solve for the density:

$$\rho = \frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)^2 . \tag{28}$$

Since $a(t) \propto t^{1/2}$, then we can conclude $\dot{a}/a = 1/(2t)$, so that

$$\rho(t) = \frac{3}{32\pi t^2} . \tag{29}$$

5 Hartle 18.24: The Einstein Static Universe.

Consider a closed ($k = +1$) FRW model containing a matter density ρ_m, a vacuum energy density corresponding to a positive cosmological constant Λ, and no radiation. *Comment:* This is the Einstein static universe for which Einstein originally introduced the cosmological universe.

5.a Show that for a given value of Λ, there is a critical value of ρ_m for which the scale factor does not change with time. Find this value.

In a universe with only mass and vacuum energy, the potential given by (Hartle 18.78) becomes

$$U_{\text{eff}}(\tilde{a}) = -\frac{1}{2}\left(\Omega_v \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}}\right) \ , \tag{30}$$

where

$$\tilde{a}(t) \equiv a(t)/a(t_0) \ . \tag{31}$$

Let us extremize the potential with respect to $\tilde{a}(t)$:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tilde{a}} U_{\text{eff}}(\tilde{a}) = -\frac{1}{2}\left(2\Omega_v \tilde{a} - \frac{\Omega_m}{\tilde{a}^2}\right) \ , \tag{32}$$

rearranging:

$$\frac{\Omega_m}{\tilde{a}^2} = 2\Omega_v \tilde{a} \quad \Rightarrow \quad \frac{\rho_m(t_0)}{\tilde{a}^3} = 2\rho_v(t_0) \ , \tag{33}$$

using (Hartle 18.33). Now if the scale factor does not change with time, then $\tilde{a} = 1$ for all times, and:

$$\rho_m = 2\rho_v = \frac{\Lambda}{4\pi} \ , \tag{34}$$

by (Hartle 18.28).

5.b What is the spatial volume for this universe in terms of Λ?

Using (Hartle 18.55) the volume of a universe is given by

$$V = 2\pi^2 a^3(t) \ , \tag{35}$$

so all that remains is to find $a(t)$ in terms of Λ. From the previous part, we know $\mathrm{d}\tilde{a}/\mathrm{d}\tilde{t} = 0$ (with $\tilde{t} = H_0 t$). This implies the Friedman equation (Hartle 18.77) becomes

$$2U_{\text{eff}} = \Omega_c \ , \tag{36}$$

in this universe. We will use the expression for Ω_c given by (Hartle 18.75), using $k = +1$:

$$-\left(\Omega_v \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}}\right) = -\frac{1}{H_0^2 a_0^2} \ , \tag{37}$$

with the shorthand $a_0 \equiv a(t_0)$. Minor rearrangement yields

$$-1 = -H_0^2 a_0^2 \left(\Omega_v \frac{a^2}{a_0^2} + \Omega_m \frac{a_0}{a}\right) = -H_0^2 \left(\Omega_v a^2 + \Omega_m \frac{a_0^3}{a}\right) = -H_0^2 \left(\Omega_v + \Omega_m \frac{a_0^3}{a^3}\right) a^2 \ , \tag{38}$$

now we use the fact that $a(t)/a_0 = 1$ in this universe to see

$$1 = \pm H_0 a\sqrt{\Omega_m + \Omega_v} = \pm 3H_0 a\Omega_v = \pm\sqrt{3}H_0 a\sqrt{\frac{\rho_v}{\rho_{\rm crit}}} \ , \tag{39}$$

after taking the square-root and using the result from the previous part. Now using (Hartle 18.28) and (Hartle 18.32), we see

$$1 = \pm\sqrt{3}\sqrt{\frac{8\pi\rho_{\rm crit}}{3}}a\sqrt{\frac{\Lambda/8\pi}{\rho_{\rm crit}}} = \pm a\sqrt{\Lambda} \ , \tag{40}$$

thus

$$a = \frac{1}{\sqrt{\Lambda}} \ , \tag{41}$$

after selecting a sign. Thus the universe's volume is

$$V = 2\pi^2\Lambda^{-3/2} \ . \tag{42}$$

5.c If ρ_m differs slightly from this value, the scale factor will vary in time. Does the evolution remain close to the static universe or diverge from it?

Let us look into the effective potential for the scale factor. Using our results from the first part, and $\rho_{\rm crit} = 3H_0^2/8\pi$, we can express the potential (30) as

$$U_{\rm eff}(\tilde{a}) = -\frac{1}{2}\left(\frac{\tilde{a}^3+2}{3\tilde{a}}\right)\frac{\Lambda}{H_0^2} \ . \tag{43}$$

The above expression is plotted in figure 1 in units of Λ/H_0^2 as a function of $\tilde{a}$. It is clear to see $\tilde{a} = 1$ (denoted by the red vertical line) is a maximum of the potential, and is therefore an unstable location. Any minute departure from $\rho_m = 2\rho_v$, will cause the universe to diverge from the static case. Divergence is faster for $\rho_m < 2\rho_v$.

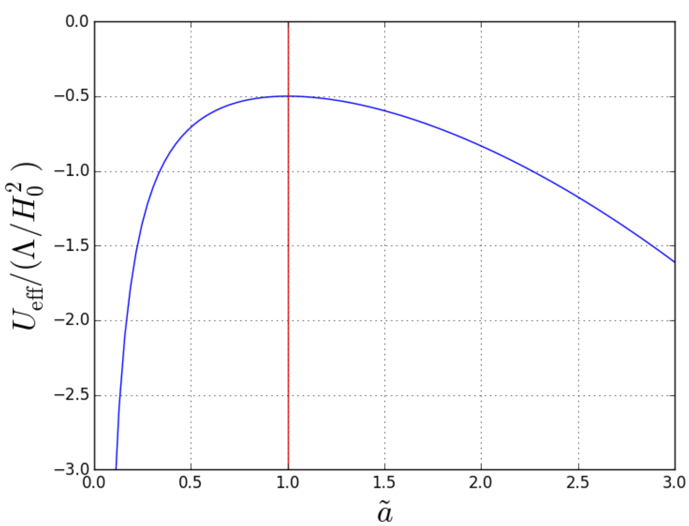

Figure 1: The effective potential for a closed universe with $\Omega_r = 0$ and a cosmological constant Λ, plotted in units of Λ/H_0^2. The red vertical line denotes the static universe: $\tilde{a} = 1$. Note that this is a maximum of the potential, and thus a departure from $\rho_m = 2\rho_v$ will result in a scale factor that does change in time.