Dylan J. Temples: Solution Set Three

Quantum Field Theory I
Quantum Field Theory and the Standard Model - M. Schwartz
October 31, 2016

Contents

1 Lorentz transformation properties.

A) A Lorentz transformation Λ^μ_ν leaves the metric tensor $g_{\mu\nu}$ invariant: ${\Lambda^\mu}_\alpha {\Lambda^\nu}_\beta g_{\mu\nu} = g_{\alpha\beta}$. Use this equation to prove that ${\Lambda_0}^0 \geq 1$ or ${\Lambda_0}^0 \leq -1$.

Using the invariance property, we see

$$g_{00} = \Lambda^\mu_0 \Lambda^\nu_0 g_{\mu\nu} \ , \tag{1}$$

or

$$\begin{aligned} 1 &= \Lambda^\mu_0 (\Lambda^0_0 g_{\mu 0} + \Lambda^1_0 g_{\mu 1} + \Lambda^2_0 g_{\mu 2} + \Lambda^3_0 g_{\mu 3}) && (2) \\ &= (\Lambda^0_0)^2 g_{00} + (\Lambda^1_0)^2 g_{11} + (\Lambda^2_0)^2 g_{22} + (\Lambda^3_0)^2 g_{33}) && (3) \\ &= (\Lambda^0_0)^2 - \left\{ (\Lambda^1_0)^2 + (\Lambda^2_0)^2 + (\Lambda^3_0)^2 \right\} \ . && (4) \end{aligned}$$

If we enforce that $\Lambda^\nu_\mu \in \mathbb{R}$, then

$$(\Lambda^1_0)^2 + (\Lambda^2_0)^2 + (\Lambda^3_0)^2 \geq 0 \ , \tag{5}$$

we can define a number $\alpha \geq 0$ by

$$\alpha \equiv (\Lambda^1_0)^2 + (\Lambda^2_0)^2 + (\Lambda^3_0)^2 \ . \tag{6}$$

Rearranging Equation 4, we have

$$\Lambda^0_0 = \pm\sqrt{1+\alpha} \ , \tag{7}$$

and since $\alpha \geq 0$, then $|\sqrt{1+\alpha}| \geq 1$, yielding the results

$$\Lambda^0_0 \geq +1 \quad \text{or} \quad \Lambda^0_0 \leq -1 \ . \tag{8}$$

B) Show that if two Lorentz transformations Λ_1 and Λ_2 both have ${(\Lambda_1)^0}_0 \geq 1$ and ${(\Lambda_2)^0}_0 \geq 1$, then $\Lambda_3 = \Lambda_1\Lambda_2$ also has ${(\Lambda_3)^0}_0 \geq 1$. In other words, this sign is preserved under Lorentz group action and can be used to classify Lorentz transformations.

Let us begin be defining ${\Phi^\alpha}_\beta = {\Lambda^\alpha}_\mu {\Omega^\mu}_\beta$, where Φ, Λ, Ω are Lorentz transformations, for notational simplicity. Consider the component:

$${\Phi^0}_0 = {\Lambda^0}_\mu {\Omega^\mu}_0 = {\Lambda^0}_0 {\Omega^0}_0 + {\Lambda^0}_1 {\Omega^1}_0 + {\Lambda^0}_2 {\Omega^2}_0 + {\Lambda^0}_3 {\Omega^3}_0 \ , \tag{9}$$

where we note that in this case ${\Lambda^0}_0 {\Omega^0}_0 \geq 1$, so

$${\Phi^0}_0 \geq 1 + \left\{ {\Lambda^0}_1 {\Omega^1}_0 + {\Lambda^0}_2 {\Omega^2}_0 + {\Lambda^0}_3 {\Omega^3}_0 \right\} \ . \tag{10}$$

C) Show that if two Lorentz transformations Λ_1 and Λ_2 both have $\det(\Lambda_1) > 0$ and $\det(\Lambda_2) > 0$, then $\Lambda_3 = \Lambda_1\Lambda_2$ also has $\det(\Lambda_3) > 0$. In other words, this sign is preserved under Lorentz group action and can be used to classify Lorentz transformations.

Consider the determinant of Λ_3:

$$\text{Det}[\Lambda_3] = \text{Det}[\Lambda_1\Lambda_2] = \text{Det}[\Lambda_1]\text{Det}[\Lambda_2] \ , \tag{11}$$

so clearly if $\text{Det}(\Lambda_1) > 0$ and $\text{Det}(\Lambda_2) > 0$, then $\text{Det}(\Lambda_3) > 0$.

D) Show all Lorentz transformations with $\det(\Lambda) > 0$ and $\Lambda^0{}_0 \geq 1$ form a subgroup of the Lorentz group.

The first requirement for a subgroup is that it contains the identity. Consider the Lorentz transform that has no action:

$$\Lambda^\mu{}_\nu = \mathbb{1}_{4\times4} \ , \tag{12}$$

which has $\det(\Lambda) > 0$ and $\Lambda^0{}_0 \geq 1$ - and is the identity. Additionally, from part (B), we know that all members of the Lorentz group with $\Lambda^0{}_0 \geq 1$ multiplied to another will remain with $\Lambda^0{}_0 \geq 1$. Similarly, from part (C), multiplication of members with $\det(\Lambda) > 0$ with other members with the same property, map to Lorentz group members with $\det(\Lambda) > 0$. From this we see that all actions on members with $\det(\Lambda) > 0$ and $\Lambda^0{}_0 \geq 1$ by members with this property, have the same property - thus satisfying closure. We can then define the subgroup of the Lorentz group $SO(1,3)$, with $\det(\Lambda) > 0$ and $\Lambda^0{}_0 \geq 1$ because it contains the identity, and no actions between members of the subgroup map to an object outside this subgroup - therefore meeting the necessary criteria for a subgroup.

2 Schwartz 10.2.

In this problem you will construct the finite-dimensional irreducible representations of $SU(2)$. By definition, such a representation is a set of three $n \times n$ matrices τ_1, τ_2, and τ_3 satisfying the algebra of the Pauli matrices $[\tau_i, \tau_j] = i\epsilon_{ijk}\tau_k$. It is also helpful to define linear combinations $\tau^\pm = \tau_1 \pm i\tau_2$.

A) In any such representation we can diagonalize τ_3. Its eigenvectors are n complex vectors V_j with $\tau_3 V_j = \lambda_j V_j$. Show that $\tau^+ V_j$ and $\tau^- V_j$ either vanish or are eigenstates of τ_3 with eigenvalues $\lambda_j + 1$ and $\lambda_j - 1$, respectively.

We begin by investigating the action of $\tau^\pm$ on the given eigenvalue equation:

$$\begin{aligned} \lambda_j \tau^\pm V_j = \tau_3^\tau V_j &= (\tau_1 \pm i\tau_2)\tau_3 V_j = \tau_1\tau_3 V_j \pm i\tau_2\tau_3 V_j && (13)\\ &= \tau_3\tau_1 V_j + [\tau_1, \tau_3]V_j \pm \{i\tau_3\tau_2 V_j + [\tau_2, \tau_3]V_j\} && (14)\\ &= \tau_3(\tau_1 \pm i\tau_2)V_j + ([\tau_1, \tau_3] \pm i[\tau_2, \tau_3])\, V_j \ , && (15)\end{aligned}$$

and using the Pauli algebra, we have

$$[\tau_1, \tau_3] \pm i[\tau_2, \tau_3] = i(-1)\tau_2 \pm i^2(+1)\tau_1 = \mp\tau_1 - i\tau_2 = \mp(\tau_1 \pm i\tau_2) = \mp\tau^\pm \ . \tag{16}$$

If we insert this result, Equation 15 becomes

$$\lambda_j \tau^\pm V_j = \tau_3\tau^\pm V_j \mp \tau^\pm V_j = (\tau_3 \mp 1)\tau^\pm V_j \ , \tag{17}$$

and isolating the term with τ_3 yields

$$\lambda_j \tau^\pm V_j \pm \tau^\pm V_j = \tau_3 \tau^\pm V_j \quad \Rightarrow \quad \tau_3(\tau^\pm V_j) = (\lambda_j \pm 1)(\tau^\pm V_j) \ . \tag{18}$$

This shows that $\tau^\pm V_j$ are eigenstates of τ_3, and they have eigenvalues $\lambda_j \pm 1$.

B) Prove that exactly one of the eigenstates V_{max} of τ_3 must satisfy $\tau^+ V_{\text{max}} = 0$. The eigenvalue $\lambda_{\text{max}} = J$ of V_{max} is known as the spin. Similarly, there will be an eigenvector V_{min} of τ_3 with $\tau^- V_{\text{min}} = 0$.

We know there are a finite number of eigenvalues, due to the dimensionalty of the system. Since the action of $\tau^\pm$ increases or decreases the eigenvalue by unity, this cannot go on forever. This implies there is some V_{max} for which

$$\begin{aligned} \tau^+ V_{\text{max}} &= 0 && (19)\\ \tau^3 V_{\text{max}} &= J V_{\text{max}} \ , && (20)\end{aligned}$$

as well as some V_{min} for which

$$\begin{aligned} \tau^- V_{\text{min}} &= 0 && (21)\\ \tau^3 V_{\text{min}} &= -J V_{\text{min}} \ . && (22)\end{aligned}$$

C) Since there are a finite number of eigenvectors, $V_{\text{min}} = (\tau^-)^N V_{\text{max}}$ for some integer N. Prove that $N = 2J$ so that $n = 2J + 1$.

Given the maximum and minimum eigenvalues, we can show there are

$$J - (-J) = 2J \ , \tag{23}$$

steps, each of size unity so we can define $N = 2J$ as the number of possible steps. This implies

$$V_{\text{max}} = (\tau^+)^N V_{\text{min}} \tag{24}$$

$$V_{\text{min}} = (\tau^-)^N V_{\text{min}} \ . \tag{25}$$

Furthermore from the dimensionality of the space, we know there must be n eigenvalues/vectors. We can count the eigenvectors:

$$\begin{aligned} V_{\text{min}} &= (\tau^+)^0 V_{\text{min}} \\ V_1 &= (\tau^+)^1 V_{\text{min}} \\ V_2 &= (\tau^+)^2 V_{\text{min}} \\ &\cdots \\ V_N &= (\tau^+)^{2J} V_{\text{min}} \ , \end{aligned}$$

and we see that $n = N + 1 = 2J + 1$.

3 Continuity equations and the Dirac equation.

In non-relativistic quantum mechanics, the Schrodinger equation implies a continuity equation of the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \ , \tag{26}$$

where $\rho = |\psi|^2$ is interpreted as the probability density.

A) Derive a similar continuity equation for the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi = 0 \ . \tag{27}$$

Write down ρ and $\mathbf{j}$ explicitly. Explain what would go wrong if you were to interpret ρ as the probability density.

Consider multiplying the KG equation on the left by ψ^*, and multiplying the complex conjugate of the KG equation on the left by ψ:

$$0 = \psi^*(\partial_\mu \partial^\mu + m^2)\psi \tag{28}$$
$$0 = \psi(\partial_\mu \partial^\mu + m^2)\psi^* \ , \tag{29}$$

the subtracting the second from the first to obtain

$$0 = \psi^* \partial_\mu \partial^\mu \psi - \psi \partial_\mu \partial^\mu \psi^* + m^2(\psi^*\psi - \psi\psi^*) \ , \tag{30}$$

and the last term vanishes. If we write $\partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$, then the above equation can be rewritten

$$0 = \psi^* \partial_t^2 \psi - \psi \partial_t^2 \psi^* + (\psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi) \ . \tag{31}$$

Here we can note:

$$\partial_t(\psi^* \partial_t \psi - \psi \partial_t \psi^*) = \partial_t \psi^* \partial_t \psi + \psi^* \partial_t^2 \psi - \partial_t \psi \partial_t \psi^* - \psi \partial_t^2 \psi^* = \psi^* \partial_t^2 \psi - \psi \partial_t^2 \psi^* \ , \tag{32}$$

and

$$\boldsymbol{\nabla} \cdot [\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^*] = \boldsymbol{\nabla} \psi^* \cdot \boldsymbol{\nabla} \psi + \psi^* \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \psi - \boldsymbol{\nabla} \psi \cdot \boldsymbol{\nabla} \psi^* - \psi \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \psi^* \tag{33}$$
$$= \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \ , \tag{34}$$

which allows us to rewrite Equation 31 as

$$0 = \partial_t \left[\psi^* \partial_t \psi - \psi \partial_t \psi^*\right] - \boldsymbol{\nabla} \cdot \left[\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^*\right] \tag{35}$$
$$= \partial_t \left[\psi^* \partial_t \psi - \psi \partial_t \psi^*\right] + \boldsymbol{\nabla} \cdot \left[\psi \boldsymbol{\nabla} \psi^* - \psi^* \boldsymbol{\nabla} \psi\right] \ . \tag{36}$$

From this, we identify

$$\rho = \psi^* \partial_t \psi - \psi \partial_t \psi^* \tag{37}$$
$$\mathbf{j} = \psi \boldsymbol{\nabla} \psi^* - \psi^* \boldsymbol{\nabla} \psi \ , \tag{38}$$

but we see that ρ is not guaranteed to be real, and thus the probability density may have an imaginary component.

B) Dirac realized in 1928 while staring at a fireplace in St. John's College, Cambridge, that the sickness in (A) is the result of an equation of motion that is second order in time derivative. He thus proposed instead

$$i\frac{\partial \psi}{\partial t} = H_D \psi \ . \tag{39}$$

Explain why the Hamiltonian H_D here must be first order in spatial derivatives and contain only terms linear in the mass m.

In relativity the time derivative has units of mass, so that the right side must have terms which have units of mass. The spatial derivatives have units of mass, and obviously so does the mass. Therefore only first derivatives and terms linear in mass can be in the Dirac Hamiltonian.

C) Dirac then guessed a general form of H_D

$$H_D = -ia_i \frac{\partial}{\partial x^i} + a_4 m \ . \tag{40}$$

H_D must be such that when applying Equation 39 twice one recovers the Klein-Gordon equation. (After all, $E^2 - |\mathbf{p}|^2 = m^2$!) First show that the set $\{a_i, i = 1,2,3,4\}$ cannot be pure numbers, then derive the conditions a_i must satisfy to produce Klein-Gordon equation. Rewrite $\gamma^0 = a_4$ and $\gamma^i = a_4 a_i$ and re-express your conditions in terms of γ^μ.

Consider the action of the Hamiltonian twice on a field ψ:

$$\begin{aligned} H_D^2\psi &= (-ia_j\partial_j + a_4 m)(-ia_i\partial_i\psi + a_4 m\psi) && (41)\\ &= -a_j a_i \partial_j\partial_i\psi + (a_4)^2 m^2\psi - ia_j a_4 m\partial_j\psi - ia_4 a_i m\partial_i\psi && (42)\\ &= -a_j a_i \partial_j\partial_i\psi + (a_4)^2 m^2\psi - im(a_j a_4\partial_j + a_4 a_i\partial_i)\psi \ , && (43) \end{aligned}$$

but we must have that

$$H^2\psi = p^2\psi + m^2\psi \ , \tag{44}$$

so

$$\begin{aligned} \mathbb{1} &= (a_4)^2 && (45)\\ 0 &= a_j a_4 \partial_j + a_4 a_i \partial_i \ , && (46) \end{aligned}$$

but the second constraint can be modified by renaming the dummy indices: $(a_i a_4 + a_4 a_i)\partial_i$, yielding the constraint

$$\{a_i, a_4\} = 0 \ , \tag{47}$$

and so the a_i cannot be pure numbers because pure numbers do not anticommute. Additionally, to reproduce the Klein-Gordon equation, it must be that

$$a_j a_i \partial_j \partial_i = \partial^j \partial_j \ , \tag{48}$$

so that $a_j a_i$ must be a symmetric matrix, which can be expressed as the spatial components of the metric. The constraints can then be summarized as

$$\mathbb{1} = a_4^2 \tag{49}$$
$$0 = \{a_i, a_4\} \tag{50}$$
$$g^{ij} = a_j a_i \ . \tag{51}$$

Now consider the anticommutator

$$\{a_i, a_j\} = a_i a_j + a_j a_i = g^{ij} + g^{ji} = 2g^{ij} \ , \tag{52}$$

and thus

$$\{a_4 a_i, a_4 a_j\} = a_4 a_i a_4 a_j + a_4 a_j a_4 a_i = -(a_4)^2 a_i a_j - (a_4)^2 a_j a_i = -\mathbb{1}\{a_i, a_j\} = -\mathbb{1}(2g^{ij}) \tag{53}$$

using the gamma matrices, we have

$$\{\gamma^i, \gamma^j\} = -\mathbb{1}(2g^{ij}) \ , \tag{54}$$

from which we extrapolate

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{55}$$
$$(\gamma^0)^2 = \mathbb{1} \quad \text{and} \quad (\gamma^i)^2 = -\mathbb{1} \ , \tag{56}$$

where the minus sign disappears when we consider the spatial and temporal components due to the sign differences in the metric.

D) What is the continuity equation following from the Dirac equation? Give ρ and $\mathbf{j}$ explicitly. Can you interpret ρ as the probability density?

Starting from Schwartz equation 10.63, we multiply on the left with $\bar{\psi}$, and multiply the adjoint[1] of Schwartz equation 10.63 on the right with ψ to obtain

$$0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{57}$$
$$0 = \bar{\psi}(i\partial_\mu \gamma^\mu - m^\star)\psi \ , \tag{58}$$

and we can assume $m = -m^\star$ because we have previously shown the pure phase of m is unphysical and does not impact the solution. We now subtract the second from the first, yielding

$$0 = i\bar{\psi}\gamma^\mu \partial_\mu \psi - i\bar{\psi}\partial_\mu \gamma^\mu \psi + im(\bar{\psi}\psi - \bar{\psi}\psi) \ , \tag{59}$$

where the mass terms vanish. Dividing out the imaginary unit, we find

$$0 = \bar{\psi}\gamma^\mu \partial_\mu \psi + \bar{\psi}\partial_\mu \gamma^\mu \psi = \partial_\mu \left(\bar{\psi}\gamma^\mu \psi\right) \ . \tag{60}$$

We then define the four-current $j^\mu = \bar{\psi}\gamma^\mu\psi$, such that

$$\rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger \gamma^0 \gamma^0 \psi = \psi^\dagger \psi \tag{61}$$
$$\mathbf{j} = j^i = \bar{\psi}(\gamma^i)\psi = \psi^\dagger \gamma^0 \gamma^i \psi \ , \tag{62}$$

and we are free to interpret ρ as the probability density in this case.

[1]
$$(i\gamma^\mu \partial_\mu - m)^\dagger = -i(\gamma^\mu \partial_\mu) - m^\star = -i(-\partial_\mu \gamma^\mu) + m = i\partial_\mu \gamma^\mu + m \ .$$

4 Dirac spinors.

In class we defined a Dirac spinor ψ in the chiral basis:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{63}$$

where ψ_L and ψ_R are Weyl spinors transforming under the Lorentz group according to Eq. (3.37) in Peskin and Schroeder. The Dirac Lagrangian is then written as

$$\mathcal{L} = i\psi^\dagger \partial_0 \psi + i\psi^\dagger \boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\psi - m\psi^\dagger \beta \psi \ , \tag{64}$$

where $\boldsymbol{\alpha}$ and β are 4×4 matrices.

A) Write out $\boldsymbol{\alpha}$ and β explicitly and show they satisfy the same conditions for $\{a_i, i = 1, 2, 3, 4\}$ you work out in Problem 2(C) if you identify $\alpha^i = a^i, i = 1, 2, 3$ and $\beta = a_4$. Then go to a different basis, the Dirac basis, for a Dirac spinor

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} \tag{65}$$

and write out $\boldsymbol{\alpha}$ and β in this basis. Show that in this basis $\boldsymbol{\alpha}$ and β still satisfy the same conditions as in the chiral basis.

The form of the Dirac Lagrangian derived in class is

$$\mathcal{L} = i\bar{\psi}\partial_\mu \gamma^\mu \psi - m\bar{\psi}\psi \ , \tag{66}$$

where we can insert $\bar{\psi} = \psi^\dagger \gamma^0$, and expand the derivative into temporal and spatial terms

$$\begin{aligned} \mathcal{L} &= i\psi^\dagger \gamma^0 \partial_0 \gamma^0 \psi - i\psi^\dagger \gamma^0 \partial_i \gamma^i \psi - m\psi^\dagger \gamma^0 \psi & (67)\\ &= i\psi^\dagger (\gamma^0)^2 \partial_0 \psi - i\psi^\dagger \gamma^0 \gamma^i \partial_i \psi - m\psi^\dagger \gamma^0 \psi & (68)\\ &= i\psi^\dagger \partial_0 \psi - i\psi^\dagger \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla}\psi - m\psi^\dagger \gamma^0 \psi \ . & (69) \end{aligned}$$

Comparing this to Equation 64, we see

$$\begin{aligned} \alpha^i &= -\gamma^0 \gamma^i = \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} & (70)\\ \beta &= \gamma^0 = \begin{pmatrix} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & \end{pmatrix} \ . & (71) \end{aligned}$$

From the previous problem, we see

$$\beta^2 = (\gamma^0)^2 = \begin{pmatrix} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & \end{pmatrix} \begin{pmatrix} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{2\times2} & \\ & \mathbb{1}_{2\times2} \end{pmatrix} = \mathbb{1} \ , \tag{72}$$

and

$$\alpha^2 = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} = \begin{pmatrix} (\sigma^i)^2 & \\ & (-\sigma^i)^2 \end{pmatrix} = \begin{pmatrix} -\mathbb{1}_{2\times2} & \\ & -\mathbb{1}_{2\times2} \end{pmatrix} = -\mathbb{1} \ , \tag{73}$$

and therefore α, β obey the same conditions from problem 3. The transformation from the chiral basis to the Dirac basis is given by

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} , \tag{74}$$

so that

$$\alpha_i \to T\alpha_i T^T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_i & \\ & -\sigma_i \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_i & -\sigma_i \\ -\sigma_i & -\sigma_i \end{pmatrix} \tag{75}$$

$$= - \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{76}$$

$$\beta \to T\beta T^T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{77}$$

B) Stay in the chiral basis, work out how a Dirac spinor transform under rotation and boost, respectively. Write out the generators for the rotation and the boost explicitly, from which deduce the generators $S^{\mu\nu}$ for the Lorentz group. Moreover, show that

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] \tag{78}$$

where the Dirac gamma matrices are defined in Problem 2(C).

In class we showed how the Weyl spinors transform under boosts and rotations:

$$\text{Boost :} \quad \psi_L \to e^{\beta \cdot \frac{\sigma}{2}} \psi_L \quad \text{and} \quad \psi_R \to e^{-\beta \cdot \frac{\sigma}{2}} \psi_R \tag{79}$$

$$\text{Rotation :} \quad \psi_L \to e^{-i\theta \cdot \frac{\sigma}{2}} \psi_L \quad \text{and} \quad \psi_R \to e^{-\theta \cdot \frac{\sigma}{2}} \psi_R . \tag{80}$$

Since the different chiralities transform differently, we can write the boosts as

$$S^{0i} = \frac{i}{2} \begin{pmatrix} \sigma_i & \\ & -\sigma_i \end{pmatrix} \tag{81}$$

such that

$$\psi \to e^{-iS^{0i}} \psi , \tag{82}$$

with

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} . \tag{83}$$

Similarly we write

$$S^{ij} = \frac{i}{2} \epsilon_{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} , \tag{84}$$

because they transform the same under a spatial rotation. The commutator of two gamma matrices is

$$[\gamma^i, \gamma^j] = \gamma^i\gamma^j - \gamma^j\gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \begin{pmatrix} & \sigma_j \\ -\sigma_j & \end{pmatrix} - \begin{pmatrix} & \sigma_j \\ -\sigma_j & \end{pmatrix} \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \tag{85}$$

$$= \begin{pmatrix} -\sigma_i\sigma_j & \\ & -\sigma_i\sigma_j \end{pmatrix} - \begin{pmatrix} -\sigma_j\sigma_i & \\ & -\sigma_j\sigma_i \end{pmatrix} = - \begin{pmatrix} [\sigma_i, \sigma_j] & \\ & [\sigma_i, \sigma_j] \end{pmatrix} \tag{86}$$

$$= -2\epsilon_{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} . \tag{87}$$

Comparing this to Equation 84, we see

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] \ . \tag{88}$$

Now consider the commutator

$$[\gamma^i, \gamma^0] = \gamma^i\gamma^0 - \gamma^0\gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix}\begin{pmatrix} & \sigma_0 \\ \sigma_0 & \end{pmatrix} - \begin{pmatrix} & \sigma_0 \\ \sigma_0 & \end{pmatrix}\begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \tag{89}$$

$$= \begin{pmatrix} \sigma_i & \\ & -\sigma_i \end{pmatrix} - \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix} = 2\begin{pmatrix} \sigma_i & \\ & -\sigma_i \end{pmatrix} \ , \tag{90}$$

from which we see

$$S^{0i} = \frac{i}{4}[\gamma^i, \gamma^0] \ . \tag{91}$$

If we elevate $i, j \to \mu, \nu$, we acheive the result

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \ . \tag{92}$$

5 Schwartz 10.5.

Supersymmetry.

A) Show that the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\star \partial^\mu \phi + \chi^\dagger i\bar{\sigma}\partial\chi + F^\star F + m\phi F + \frac{i}{2} m\chi^T \sigma^2 \chi + \text{h.c.} \tag{93}$$

is invariant under

$$\delta\phi = -i\epsilon^T \sigma^2 \chi \ , \tag{94}$$

$$\delta\chi = \epsilon F + \sigma^\mu \partial_\mu \phi \sigma^2 \epsilon^\star \ , \tag{95}$$

$$\delta F = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi \ , \tag{96}$$

where ϵ is an infinitesimal spinor, χ is a spinor, and F and ϕ are scalars. All spinors anti-commute. σ^2 is the second Pauli spin matrix.

Let us begin by definining

$$T_1 = \partial_\mu \phi^\star \partial^\mu \phi \tag{97}$$

$$T_2 = \chi^\dagger i\bar{\sigma}\partial\chi \tag{98}$$

$$T_3 = F^\star F \tag{99}$$

$$T_4 = m\phi F \tag{100}$$

$$T_5 = \frac{i}{2} m_\chi^T \sigma^2 \chi \ , \tag{101}$$

such that the Lagrangian is

$$\mathcal{L} = T_1 + T_2 + T_3 + T_4 + T_5 + T_1^\dagger + T_2^\dagger + T_3^\dagger + T_4^\dagger + T_5^\dagger \ . \tag{102}$$

To which we make the transformation $\phi \to \phi + \delta\phi$, $\chi \to \delta\chi$, and $F \to F + \delta F$, with

$$\delta\phi = -i(\epsilon_1 \ \epsilon_2) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{103}$$

$$= \epsilon_2\chi_1 - \epsilon_1\chi_2 \tag{104}$$

$$\delta F = -i(\epsilon_1^\star \ \epsilon_2^\star)\bar{\sigma}^\mu \partial_\mu \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{105}$$

$$= -i\epsilon_1^\star \left[\bar{\sigma}^\mu \partial\mu\chi\right]_1 - i\epsilon_2^\star \left[\bar{\sigma}^\mu \partial\mu\chi\right]_2 \tag{106}$$

$$\delta\chi = \epsilon F + \sigma^\mu \partial_\mu \phi \sigma^2 \epsilon^\star \ . \tag{107}$$

The term T_1, to linear order in ϵ becomes

$$T_1 \to \partial_\mu \phi^\star \partial_\mu \phi + \partial_\mu \delta\phi^\star \partial^\mu \phi + \partial_\mu \phi^\star \partial^\mu \delta\phi = T_1 + \partial_\mu \phi^\star \partial^\mu \delta\phi + \partial_\mu \delta\phi^\star \partial^\mu \phi \tag{108}$$

$$T_1 \to T_1 + \tau_1 + \tau_2 \ , \tag{109}$$

where

$$\delta\phi^\star = \left(-i\epsilon^T \sigma^2 \chi\right)^\star = (\epsilon_2\chi_1 - \epsilon_1\chi_2)^\star = -\left(\epsilon_1^\star \chi_2^\star - \epsilon_2^\star \chi_1^\star\right) = -\left(-i\epsilon^{\star T} \sigma^{2\star} \chi^\star\right) \tag{110}$$

$$= -(i\epsilon^{T\star} \sigma^2 \chi^\star) \ , \tag{111}$$

from which we see

$$\tau_1 = -i\epsilon^T \sigma^2 \partial_\mu \phi^\star \partial^\mu \chi \tag{112}$$
$$\tau_2 = -i\epsilon^{\star T} \sigma^2 \partial_\mu \chi^\star \partial^\mu \phi \ . \tag{113}$$

Additionally, for the term T_2, to linear order in ϵ, we have

$$T_2 \to (\chi^\dagger + \delta\chi^\dagger) i\bar{\sigma}^\mu \partial_\mu (\chi + \delta\chi) = T_2 + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \delta\chi + \delta\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi \ , \tag{114}$$

where

$$\delta\chi^\dagger = \epsilon^\dagger F^\star + \epsilon^T \sigma^2 \sigma^\mu \partial_\mu \phi^\star \ , \tag{115}$$

because the Pauli matrices are Hermitian. The derivatives of the variation of χ and its Hermitian conjugate are

$$\partial_\mu \chi = \epsilon \partial_\mu F + \sigma^\mu \partial_\mu^2 \phi \sigma^2 \epsilon^\star \tag{116}$$
$$\partial_\mu \chi^\dagger = \epsilon^\dagger \partial_\mu F^\star + \epsilon^T \sigma^2 \sigma^\mu \partial_\mu^2 \phi^\star \ , \tag{117}$$

so we see

$$T_2 \to T_2 + \chi^\dagger i\bar{\sigma}^\mu \left[\epsilon \partial_\mu F + \sigma^\mu \partial_\mu^2 \phi \sigma^2 \epsilon^\star\right] + \left[\epsilon^\dagger F^\star + \epsilon^T \sigma^2 \sigma^\mu \partial_\mu \phi^\star\right] i\bar{\sigma}^\mu \partial_\mu \chi \tag{118}$$
$$\to T_2 + i\chi^\dagger \bar{\sigma}^\mu \partial_\mu F \epsilon + i\chi^\dagger \bar{\sigma}^\mu \sigma^\nu \sigma^2 \epsilon^\star \partial_\mu \partial_\nu \phi + iF^\star \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\epsilon^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu \partial_\nu \phi^\star \partial_\mu \chi \ , \tag{119}$$

which allows us to define

$$\tau_3 = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu F \epsilon \tag{120}$$
$$\tau_4 = i\chi^\dagger \bar{\sigma}^\mu \sigma^\nu \sigma^2 \epsilon^\star \partial_\mu \partial_\nu \phi \tag{121}$$
$$\tau_5 = iF^\star \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi \tag{122}$$
$$\tau_6 = i\epsilon^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu \partial_\nu \phi^\star \partial_\mu \chi \ . \tag{123}$$

To linear order in ϵ, we have

$$T_3 \to T_3 + F^\star \delta F + \delta F^\star F = T_3 + \tau_7 + \tau_8 \ , \tag{124}$$

where

$$\delta F^\star = \left(-i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi\right)^\star = -(+i)\epsilon^T \bar{\sigma}^{\mu\star} \partial_\mu \chi^* \ , \tag{125}$$

where the overall negative sign comes about in a similar way as $\delta\phi^\star$. We see that

$$\tau_7 = -iF^\star \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi \tag{126}$$
$$\tau_8 = -iF\epsilon^T \bar{\sigma}^{\mu\star} \partial_\mu \chi^\star \ . \tag{127}$$

For the fourth term, we have, to linear order in ϵ:

$$T_4 \to T_4 + m\phi\delta F + m\delta\phi F = T_4 - im\phi\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi - im\epsilon^T \sigma^2 \chi F \ , \tag{128}$$

and we define

$$\tau_9 = -im\phi\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi \tag{129}$$
$$\tau_{10} = -im\epsilon^T \sigma^2 \chi F \ , \tag{130}$$

and for the final term, we have

$$T_5 \to \frac{i}{2}m\left[\chi^T\sigma^2\chi + \delta\chi^T\sigma^2\chi + \chi^T\sigma^2\delta\chi\right] \ . \tag{131}$$

Using

$$\delta\chi^T = \epsilon^T F + \epsilon^\dagger\sigma^2\sigma^\mu\partial_\mu\phi \ , \tag{132}$$

the fifth term becomes

$$T_5 \to T_5 + \frac{i}{2}m\left[\epsilon^T F + \epsilon^\dagger\sigma^2\sigma^\mu\partial_\mu\phi\right]\sigma^2\chi + \frac{i}{2}m\chi^T\sigma^2\left[\epsilon F + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^\star\right] \ , \tag{133}$$

from which we define

$$\tau_{11} \equiv \frac{i}{2}m\epsilon^T F\sigma^2\chi \tag{134}$$

$$\tau_{12} \equiv \frac{i}{2}m\epsilon^\dagger\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\chi \tag{135}$$

$$\tau_{13} \equiv \frac{i}{2}m\chi^T\sigma^2\epsilon F \tag{136}$$

$$\tau_{14} \equiv \frac{i}{2}m\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^\star \ . \tag{137}$$

We can now argue that $\sum_i \tau_i = 0$. Consider $\tau_1 + \tau_6$: the factor in τ_6

$$\sigma^\nu\bar{\sigma}^\mu\partial_\nu\phi^\star\partial_\mu\chi = \partial_\mu\chi^\star\partial^\mu\phi \ , \tag{138}$$

using the anticommutators for the σ matrices. Comparing the remaining factors, we see τ_1 and τ_6 sum to zero. Similarly for τ_2 and τ_4, from the anticommutation relations, we can write

$$\tau_4 = -i\chi^\dagger\sigma^2\epsilon^\star\partial_\mu\partial^\mu\phi \ , \tag{139}$$

which we add to τ_2 to obtain

$$\tau_2 + \tau_4 = -i\epsilon^{\star T}\sigma^2\partial_\mu\chi^\star\partial^\mu\phi - i\chi^\dagger\sigma^2\epsilon^\star\partial_\mu\partial^\mu\phi = -i\partial_\mu[\epsilon^{\star T}\sigma^2\chi^\star + \chi^\dagger\sigma^2\epsilon^\star]\partial^\mu\phi \ , \tag{140}$$

which is the divergence of a Lorentz four-vector, which does not effect the Euler-Lagrange equation - and effectively leaves the Lagranginan invariant. We see, with no manipulation, that $\tau_5 + \tau_7 = 0$. Now consider

$$\tau_3 + \tau_8 = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu F\epsilon - iF\epsilon^T\bar{\sigma}^{\mu\star}\partial_\mu\chi^\star = -i\partial_\mu[F\chi^\dagger\bar{\sigma}^{\star\mu}\epsilon] \ , \tag{141}$$

which is the divergence of a Lorentz four-vector, and is irrelevant.

B) The field F is an auxiliary field, since it has no kinetic term. A useful trick for dealing with auxiliary fields is to solve their equations of motion exactly and plug the result back into the Lagrangian. This is called integrating out a field. Integrate out F to show that ϕ and χ have the same mass.

The Lagrangian is

$$\mathcal{L} = \partial_\mu\phi^\star\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\partial\chi + F^\star F + m\phi F + m\phi^\star F^\star + \frac{im}{2}\left[\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^\star\right] \ , \tag{142}$$

for which the equations of motion for F are

$$\frac{\partial \mathcal{L}}{\partial F} = 0 \quad \Rightarrow \quad F^\star = -m\phi \tag{143}$$
$$\frac{\partial \mathcal{L}}{\partial F^\star} = 0 \quad \Rightarrow \quad F = -m\phi^\star \ , \tag{144}$$

which we insert back into the Lagrangian to yield

$$\mathcal{L} = \partial_\mu \phi^\star \partial^\mu \phi + \chi^\dagger i\bar{\sigma}\partial\chi + m^2\phi\phi^\star - m^2\phi\phi^\star - m^2\phi^\star\phi + \frac{im}{2}\left[\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^\star\right] \tag{145}$$
$$= \partial_\mu \phi^\star \partial^\mu \phi + \chi^\dagger i\bar{\sigma}\partial\chi - m^2\phi^\star\phi + \frac{im}{2}\left[\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^\star\right] \ , \tag{146}$$

thus both fields have mass terms with mass m.

C) Auxiliary fields such as F act like Lagrange multipliers. One reason to keep the auxiliary fields in the Lagrangian is because they make the symmetry transformations exact at the level of the Lagrangian. After the field has been integrated out, the symmetries are only guaranteed to hold if you use the equations of motion. Still using $\delta\phi = i\epsilon^T\sigma^2\chi$, what is the transformation of χ that makes the Lagrangian in (B) invariant, if you are allowed to use the equations of motion.

The Lagrangian transforms (to first oreder in ϵ) as

$$\begin{aligned}\mathcal{L} \to T_1 + \tau_1 + \tau_2 + T_2 + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi + \delta\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi - m^2\phi^\star\phi - m^2\phi^\star\delta\phi - m^2\delta\phi^\star\phi \\ + \frac{im}{2}\left[\chi^T\sigma^2\chi + \delta\chi^T\sigma^2\chi + \chi^T\sigma^2\delta\chi - \chi^\dagger\sigma^2\chi^\star - \delta\chi^\dagger\sigma^2\chi^\star - \chi^\dagger\sigma^2\delta\chi\right] \ ,\end{aligned}$$

and for invariance we enfore

$$\begin{aligned}0 = -i\epsilon^T\sigma^2\partial_\mu\phi^\star\partial^\mu\chi - i\epsilon^{\star T}\sigma^2\partial_\mu\chi^\star\partial^\mu\phi - m^2\phi^\star(i\epsilon^T\sigma^2\chi) - m^2(-i\epsilon^{T\star}\sigma^2\chi^\star)\phi \\ + \frac{im}{2}\left[\delta\chi^T\sigma^2\chi + \chi^T\sigma^2\delta\chi - \delta\chi^\dagger\sigma^2\chi^\star - \chi^\dagger\sigma^2\delta\chi\right] \ .\end{aligned}$$

We then need to find the equations of motion for ϕ and $\phi^\star$, which we can insert back into the Lagrangian, therefore integrating out ϕ and obtaining a Lagrangian with terms containing χ and $\delta\chi$. We insert the equations of motion into the above constraint equation, and solve for $\delta\chi$.