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Quantum Field Theory I
Quantum Field Theory and the Standard Model - M. Schwartz
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1 Schwartz 11.1.

In practice, we only rarely use the explicit representations of the Dirac matrices. Most calculations can be done using algebraic identities that depend only on $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Derive algebraically (without using an explicit representation):

A)
$$(\gamma^5)^2 = \mathbb{1}$$

Let us begin by clarifying the notation for the anticommutator: due to the dimensionality of the left- and right-hand sides, we must have

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4 \quad \Rightarrow \quad \gamma^\mu\gamma^\nu = 2g^{\mu\nu}\mathbb{1}_4 - \gamma^\nu\gamma^\mu \ . \tag{1}$$

With that out of the way, consider the square of $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$:

$$(\gamma^5)^2 = (i^2)(\gamma^0\gamma^1)\gamma^2(\gamma^3\gamma^0)\gamma^1\gamma^2\gamma^3 \ , \tag{2}$$

and let us replace the products in the parenthesis with the anticommutation relation. Since we are anticommuting different indeces, we just pick up a negative sign when we swap the order, we do this twice, so this becomes

$$(\gamma^5)^2 = (i^2)\gamma^1(\gamma^0\gamma^2)\gamma^0(\gamma^3\gamma^1)\gamma^2\gamma^3 \ , \tag{3}$$

again replacing the products in parenthesis with their anticommutator relation, we pick up two negative signs to obtain

$$(\gamma^5)^2 = (i^2)\gamma^1\gamma^2\gamma^0\gamma^0\gamma^1\gamma^3(\gamma^2\gamma^3) \ , \tag{4}$$

replacing the product in parenthesis with the anticommutator gives

$$(\gamma^5)^2 = -(i^2)\gamma^1\gamma^2(\gamma^0)^2\gamma^1(\gamma^3)^2\gamma^2 \ . \tag{5}$$

The square of a gamma matrix is easy to compute by considering the anticommutator:

$$2g^{\mu\mu}\mathbb{1}_4 = \{\gamma^\mu, \gamma^\mu\} = 2(\gamma^\mu)^2 \ , \tag{6}$$

so $(\gamma^0)^2 = \mathbb{1}_4$ and $(\gamma^i)^2 = -\mathbb{1}_4$. Inserting this result gives

$$(\gamma^5)^2 = \mathbb{1}_4(i^2)\gamma^1(\gamma^2\gamma^1)\gamma^2 \tag{7}$$
$$= -\mathbb{1}_4(i^2)(\gamma^1)^2(\gamma^2)^2 \tag{8}$$
$$= (-1)\mathbb{1}_4(-1)(-1)(-1) = \mathbb{1}_4 \ . \tag{9}$$

B)
$$\gamma_\mu \not{p} \gamma^\mu = -2\not{p}$$

Expanding the Feynman notation yields

$$\gamma_\mu \not{p} \gamma^\mu = \gamma_\mu\gamma^\nu p_\nu\gamma^\mu = \gamma_\mu\gamma^\nu p_\nu\gamma^\mu = (g_{\mu\sigma}\gamma^\sigma)\gamma^\nu p_\nu\gamma^\mu = (g_{\mu\sigma}\gamma^\sigma)p_\nu\gamma^\nu\gamma^\mu \ , \tag{10}$$

and replacing the last two gamma matrices with their anticommutator gives

$$\gamma_\mu \not{p} \gamma^\mu = (g_{\mu\sigma}\gamma^\sigma)p_\nu(2g^{\nu\mu}\mathbb{1}_4 - \gamma^\mu\gamma^\nu) = 2g_{\mu\sigma}g^{\nu\mu}\mathbb{1}_4\gamma^\sigma p_\nu - g_{\mu\sigma}\gamma^\sigma p_\nu\gamma^\mu\gamma^\nu \ . \tag{11}$$

Performing some contractions simplifies this equation to

$$\gamma_\mu \not{p} \gamma^\mu = 2\mathbb{1}_4 \gamma_\mu p^\mu - \gamma_\mu p_\nu \gamma^\mu \gamma^\nu = 2\mathbb{1}_4 \gamma_\mu p^\mu - \gamma_\mu \gamma^\mu p_\nu \gamma^\nu \ . \tag{12}$$

Now we must evaluate the contraction

$$\gamma_\mu \gamma^\mu = g_{\mu\sigma} \gamma^\sigma \gamma^\mu = \frac{1}{2}(g_{\mu\sigma} + g_{\sigma\mu})\gamma^\sigma \gamma^\mu \ , \tag{13}$$

due to the symmetric nature of the metric tensor. If we distribute and relabel the dummy indeces on the second term, we find

$$\gamma_\mu \gamma^\mu = \frac{1}{2}(g_{\mu\sigma}\gamma^\sigma \gamma^\mu + g_{\mu\sigma}\gamma^\mu \gamma^\sigma) = \frac{g_{\mu\sigma}}{2}\{\gamma^\sigma, \gamma^\mu\} = \frac{1}{2} g_{\mu\sigma}(2g^{\sigma\mu}\mathbb{1}_4) = 4\mathbb{1}_4 \ , \tag{14}$$

since $g_{\alpha\beta} g^{\alpha\beta} = 4$. Using this result, we have

$$\gamma_\mu \not{p} \gamma^\mu = 2\mathbb{1}_4 \gamma_\mu p^\mu - 4\mathbb{1}_4 p_\nu \gamma^\nu = 2(\gamma^\mu p_\mu - 2\gamma^\nu p_\nu) \ , \tag{15}$$

after multiplying the identity through (acting on the gamma matrices). Relabeling the dummy indeces on the first term, and subtracting yields the result

$$\gamma_\mu \not{p} \gamma^\mu = -2\gamma^\nu p_\nu = -2\not{p} \ . \tag{16}$$

C)

$$\gamma_\mu \not{p} \not{q} \not{p} \gamma^\mu = -2\not{p}\not{q}\not{p}$$

Again, we can expand the Feynman notation, and move the vectors to the right, since they commute with the gamma matrices:

$$\gamma_\mu \not{p} \not{q} \not{p} \gamma^\mu = \gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu p_\nu q_\rho p_\sigma \ . \tag{17}$$

Now consider the quantity

$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = (g_{\mu\alpha}\gamma^\alpha)\gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = \gamma^\alpha \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\alpha \ . \tag{18}$$

Using the anticommutator for the first two matrices yields

$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = (2g^{\alpha\nu} - \gamma^\nu \gamma^\alpha)\gamma^\rho \gamma^\sigma \gamma_\alpha = 2g^{\alpha\nu}\gamma^\rho \gamma^\sigma \gamma_\alpha - \gamma^\nu \gamma^\alpha \gamma^\rho \gamma^\sigma \gamma_\alpha \ . \tag{19}$$

The first term can be written $2\gamma^\rho \gamma^\sigma \gamma^\nu$, while the second is $\gamma^\nu(\gamma^\alpha \gamma^\rho \gamma^\sigma \gamma_\alpha)$. We can express the factor in the parenthesis as

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = g_{\alpha\mu}\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\alpha = g_{\alpha\mu}\left([2g^{\mu\nu}\mathbb{1}_4 - \gamma^\nu \gamma^\mu][2g^{\rho\alpha}\mathbb{1}_4 - \gamma^\alpha \gamma^\rho]\right) \tag{20}$$

$$= g_{\alpha\mu}\left(4\mathbb{1}_4^2 g^{\mu\nu} g^{\rho\alpha} - 2\mathbb{1}_4 g^{\rho\alpha}\gamma^\nu \gamma^\mu - 2\mathbb{1}_4 g^{\mu\nu}\gamma^\alpha \gamma^\rho + \gamma^\nu \gamma^\mu \gamma^\alpha \gamma^\rho\right) \tag{21}$$

$$= 4\mathbb{1}_4 g^\nu_\alpha g^{\rho\alpha} - 2g^\rho_\mu \gamma^\nu \gamma^\mu - 2g^\nu_\alpha \gamma^\alpha \gamma^\rho + \gamma^\nu \gamma_\alpha \gamma^\alpha \gamma^\rho \ . \tag{22}$$

Note we have acted the identity on the gamma matrices in the middle two terms. If we insert the result from Equation 14, and act the identity on one of the gamma matrices, we see

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\mathbb{1}_4 g^\nu_\alpha g^{\rho\alpha} - 2g^\rho_\mu \gamma^\nu \gamma^\mu - 2g^\nu_\alpha \gamma^\alpha \gamma^\rho + 4\gamma^\nu \gamma^\rho \tag{23}$$

$$= 4g^{\nu\rho}\mathbb{1}_4 - 2\gamma^\nu \gamma^\rho - 2\gamma^\nu \gamma^\rho + 4\gamma^\nu \gamma^\rho \tag{24}$$

$$= 4g^{\nu\rho}\mathbb{1}_4 \ . \tag{25}$$

Taking this result and inserting it into Equation 19, we find

$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = 2\gamma^\rho \gamma^\sigma \gamma^\nu - \gamma^\nu (4g^{\rho\sigma} \mathbb{1}_4) = 2(\gamma^\rho \gamma^\sigma - 2g^{\rho\sigma} \mathbb{1}_4)\gamma^\nu = -2(\gamma^\sigma \gamma^\rho)\gamma^\nu \ , \tag{26}$$

which we can insert into Equation 17 to find

$$\gamma_\mu \not{p}\not{q}\not{p}\gamma^\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu p_\nu q_\rho p_\sigma = -2\gamma^\sigma p_\sigma \gamma^\rho q_\rho \gamma^\nu p_\nu = -2\not{p}\not{q}\not{p} \ . \tag{27}$$

D)

$$\{\gamma^5, \gamma^\mu\} = 0$$

Writing out the commutator yields

$$\{\gamma^5, \gamma^\mu\} = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3) \ , \tag{28}$$

consider the second term:

$$\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = (2g^{\mu 0} - \gamma^0\gamma^\mu)\gamma^1\gamma^2\gamma^3 = 2g^{\mu 0}\gamma^1\gamma^2\gamma^3 - \gamma^0\gamma^\mu\gamma^1\gamma^2\gamma^3 \ . \tag{29}$$

If we continue this process until we have anticommuted γ^μ to the end, we find

$$\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = 2g^{\mu 0}\gamma^1\gamma^2\gamma^3 - 2\gamma^0 g^{\mu 1}\gamma^2\gamma^3 + 2\gamma^0\gamma^1 g^{\mu 2}\gamma^3 - 2\gamma^0\gamma^1\gamma^2 g^{\mu 3} + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu \ . \tag{30}$$

Let us add the final term on the right-hand side to both sides:

$$\begin{aligned}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = \\ 2g^{\mu 0}\gamma^1\gamma^2\gamma^3 - 2\gamma^0 g^{\mu 1}\gamma^2\gamma^3 + 2\gamma^0\gamma^1 g^{\mu 2}\gamma^3 - 2\gamma^0\gamma^1\gamma^2 g^{\mu 3} + 2\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu \ ,\end{aligned} \tag{31}$$

multiplying by the imaginary unit yields

$$\{\gamma^5, \gamma^\mu\} = 2i\left(g^{\mu 0}\gamma^1\gamma^2\gamma^3 - \gamma^0 g^{\mu 1}\gamma^2\gamma^3 + \gamma^0\gamma^1 g^{\mu 2}\gamma^3 - \gamma^0\gamma^1\gamma^2 g^{\mu 3} + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\right) \ . \tag{32}$$

Since $\mu \in \{0, 1, 2, 3\}$ and $g^{\mu\nu}$ is diagonal, for a given choice of μ only one term containing the metric in the above equation is nonzero. Consider the case $\mu = 3$:

$$\{\gamma^5, \gamma^3\} = 2i\left(-\gamma^0\gamma^1\gamma^2 g^{33} + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\right) \tag{33}$$

$$= 2i(\gamma^0\gamma^1\gamma^2 - \gamma^0\gamma^1\gamma^2) = 0 \ , \tag{34}$$

using $(\gamma^i)^2 = -\mathbb{1}_4$. Due to the pattern of the signs of the terms with the metric and how many times the γ^i matrix must be anticommuted to get factors of $(\gamma^i)^2$, we see the above reasoning will hold for $\mu = 1, 2, 3$. We will verify explicitly for $\mu = 0$:

$$\{\gamma^5, \gamma^0\} = 2i\left(g^{00}\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\right) \tag{35}$$

$$= 2i\left(\gamma^1\gamma^2\gamma^3 + (-1)^3(\gamma^0)^2\gamma^1\gamma^2\gamma^3\right) \tag{36}$$

$$= 2i\left(\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3\right) = 0 \ , \tag{37}$$

using $(\gamma^0)^2 = +\mathbb{1}_4$. Following the logic stated above, or verifying explicity if you do not believe me, proves:

$$\{\gamma^5, \gamma^\mu\} = 0 \ . \tag{38}$$

E)

$$\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta})$$

Let us begin by anticommuting γ^ν to the right side of the expression:

$$\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = \mathrm{Tr}[\gamma^\alpha\gamma^\mu(2g^{\beta\nu}\mathbb{1}_4 - \gamma^\nu\gamma^\beta)] = 2g^{\beta\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\mu\mathbb{1}_4] - \mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\beta] \tag{39}$$

$$= 2g^{\beta\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\mu] - 2g^{\mu\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\beta] + \mathrm{Tr}[\gamma^\alpha\gamma^\nu\gamma^\mu\gamma^\beta] \tag{40}$$

$$= 2g^{\beta\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\mu] - 2g^{\mu\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\beta] + 2g^{\alpha\nu}\,\mathrm{Tr}[\gamma^\mu\gamma^\beta] - \mathrm{Tr}[\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta]\ , \tag{41}$$

where the $\mathbb{1}_4$ from the anticommutator was multiplied with a gamma matrix. Now consider the trace of the product of two gamma matrices:

$$\mathrm{Tr}[\gamma^\rho\gamma^\sigma] = \frac{1}{2}(\mathrm{Tr}[\gamma^\rho\gamma^\sigma] + \mathrm{Tr}[\gamma^\sigma\gamma^\rho]) = \frac{1}{2}\,\mathrm{Tr}[\gamma^\rho\gamma^\sigma + \gamma^\sigma\gamma^\rho] = \frac{1}{2}\,\mathrm{Tr}[\{\gamma^\rho, \gamma^\sigma\}] \tag{42}$$

$$= g^{\rho\sigma}Tr[\mathbb{1}_4] = 4g^{\rho\sigma}\ , \tag{43}$$

where we have used the fact that the trace is invariant under cyclic permutations of its arguments. Using this result, Equation 41 becomes

$$\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = 8g^{\beta\nu}g^{\alpha\mu} - 8g^{\mu\nu}g^{\alpha\beta} + 8g^{\alpha\nu}g^{\mu\beta} - \mathrm{Tr}[\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta]\ , \tag{44}$$

if we cycle the trace on the right-hand side: $\mathrm{Tr}[\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta] = \mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu]$, then move it to the right, we see the trace terms sum. Dividing by two yields the final answer:

$$\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = 4g^{\beta\nu}g^{\alpha\mu} - 4g^{\mu\nu}g^{\alpha\beta} + 4g^{\alpha\nu}g^{\mu\beta} \tag{45}$$

$$= 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta})\ . \tag{46}$$

2 Schwartz 11.4.

Show that for on-shell spinors

$$\bar{u}(q)\gamma^\mu u(p) = \bar{u}(q)\left[\frac{q^\mu + p^\mu}{2m} + i\frac{\sigma^{\mu\nu}(q_\nu - p_\nu)}{2m}\right]u(p) \ , \tag{47}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. This is known as the **Gordon Identity**. We will use this when we calculate the 1-loop correction to the elecctron's magnetic dipole moment.

Consider the anticommutator identity for the gamma matrices: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$. From the definition of the spin operator, we have that

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \ , \tag{48}$$

but we can replace the second term with the anticommutator relation:

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - (2g^{\nu\mu}\mathbb{1}_4 - \gamma^\mu\gamma^\nu)\right) = \frac{i}{2}\left(2\gamma^\mu\gamma^\nu - 2g^{\nu\mu}\mathbb{1}_4\right) \ , \tag{49}$$

so

$$i\sigma^{\mu\nu} = g^{\nu\mu}\mathbb{1}_4 - \gamma^\mu\gamma^\nu \ . \tag{50}$$

However, if we instead anticommuted the first term, we would find

$$i\sigma^{\mu\nu} = \gamma^\nu\gamma^\mu - g^{\mu\nu}\mathbb{1}_4 \ . \tag{51}$$

Let us now calculate

$$\bar{u}(q)i\sigma^{\mu\nu}(q_\nu - p_\nu)u(p) \ , \tag{52}$$

where $u(p)$ is a Dirac spinor for a fermion field with momentum p. If we distribute the Pauli matrix and replace it with both expressions we found above, this becomes

$$\bar{u}(q)i\sigma^{\mu\nu}(q_\nu - p_\nu)u(p) = \bar{u}(q)\left[(\gamma^\nu\gamma^\mu - g^{\mu\nu}\mathbb{1}_4)q_\nu - (g^{\nu\mu}\mathbb{1}_4 - \gamma^\mu\gamma^\nu)p_\nu\right]u(p) \ , \tag{53}$$

again distributing and contracting yields

$$\bar{u}(q)i\sigma^{\mu\nu}(q_\nu - p_\nu)u(p) = \bar{u}(q)\left[\gamma^\nu q_\nu\gamma^\mu - q^\mu - p^\mu + \gamma^\mu\gamma^\nu p_\nu\right]u(p) \ , \tag{54}$$

in Feynman slash notation this can be written

$$\bar{u}(q)i\sigma^{\mu\nu}(q_\nu - p_\nu)u(p) = \bar{u}(q)\left[\not{q}\gamma^\mu - (q+p)^\mu + \gamma^\mu\not{p}\right]u(p) \ . \tag{55}$$

If we insert the plane wave solutions into the Dirac equation, we find $u(p)$ satisfies the constraint (Peskin Eq. 3.46):

$$(\not{p} - m)u(p) = (\gamma^\mu p_\mu - m)u(p) = 0 \ , \tag{56}$$

and the adjoint of this is

$$\bar{u}(q)(\not{q} - m) = \bar{u}(q)(\gamma^\mu q_\mu - m) = 0 \ , \tag{57}$$

from these equations, we see

$$\not{p}u(p) = mu(p) \tag{58}$$
$$\bar{u}(q)\not{q} = \bar{u}(q)m \ . \tag{59}$$

We can insert these results into Equation 55 to see:

$$\bar{u}(q)i\sigma^{\mu\nu}(q-p)_\nu u(p) = \bar{u}(q)m\gamma^\mu u(p) - \bar{u}(q)(q+p)^\mu u(p) + \bar{u}(q)\gamma^\mu m u(p) \tag{60}$$
$$= \bar{u}(q)\left[m\gamma^\mu - (q+p)^\mu + \gamma^\mu m\right]u(p) \tag{61}$$
$$= \bar{u}(q)\left[2m\gamma^\mu - (q+p)^\mu\right]u(p) \tag{62}$$
$$0 = \bar{u}(q)\left[i\sigma^{\mu\nu}(q-p)_\nu - 2m\gamma^\mu + (q+p)^\mu\right]u(p) \ . \tag{63}$$

Bringing the term with the gamma matrix to the other side, and dividing by $2m$ yields

$$\bar{u}(q)\gamma^\mu u(p) = \bar{u}(q)\frac{1}{2m}\left[i\sigma^{\mu\nu}(q-p)_\nu + (q+p)^\mu\right]u(p) \tag{64}$$
$$= \bar{u}(q)\left[\frac{q^\mu + p^\mu}{2m} + i\frac{\sigma^{\mu\nu}(q_\nu - p_\nu)}{2m}\right]u(p) \ , \tag{65}$$

which is the Gordon Identity.

Consider the spinor

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \ , \tag{66}$$

which has the property (Peskin Eq. 3.57):

$$\bar{u}_{s'}(p)u_s(p) = 2m\xi^\dagger_{s'}\xi_s = 2m\delta_{s's} \ . \tag{67}$$

The quantity $\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p)$ can be expressed using the Gordon identity as

$$\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = \bar{u}_\sigma(p)\left[\frac{p^\mu + p^\mu}{2m} + i\frac{\sigma^{\mu\nu}(p_\nu - p_\nu)}{2m}\right]u_{\sigma'}(p) \tag{68}$$
$$= \bar{u}_\sigma(p)\left[\frac{p^\mu}{m}\right]u_{\sigma'}(p) = \left(\frac{p^\mu}{p^0}\right)2m\delta_{\sigma'\sigma} \tag{69}$$
$$= 2p^\mu\delta_{\sigma'\sigma} \ . \tag{70}$$

3 Schwartz 11.8.

Fierz rearrangement formulas (Fierz identities). It is often useful to rewrite spinor contractions in other forms to simplify formulas. Note that $P_L = \frac{1-\gamma^5}{2}$ project out the left-handed spinor from a Dirac fermion. The identities with P_L play an important role in the theory of weak interactions, which only involves left-hande spinors. **Some hints:**

1. In class we discussed there exist 16 independent matrices that can be constructed out of the Dirac matrices: $\Gamma^A = \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5\}$. Write down a basis for Γ^A such that
$$\text{Tr}[\Gamma^A\Gamma^B] = 4\delta^{AB} \ .$$
2. Write the general Fierz identity as an equation
$$(\bar{\psi}_1\Gamma^A\psi_2)(\bar{\psi}_3\Gamma^B\psi_4) = \sum_{C,D} C^{AB}_{CD}(\bar{\psi}_1\Gamma^C\psi_4)(\bar{\psi}_3\Gamma^D\psi_2) \ ,$$
with unknown coefficients C^{AB}_{CD}. Using the completeness of the 16 Γ^A matrices, show that
$$C^{AB}_{CD} = \frac{1}{16}\text{Tr}[\Gamma^C\Gamma^A\Gamma^D\Gamma^B] \ .$$
Then you can complete Problem 11.8.

Show that

A)
$$(\bar{\psi}_1\gamma^\mu P_L\psi_2)(\bar{\psi}_3\gamma^\mu P_L\psi_4) = -(\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma^\mu P_L\psi_2)$$

Let us begin by defining a set of Dirac spinors
$$\psi_i = \begin{pmatrix}\chi_i \\ \xi_i\end{pmatrix} \quad \text{so} \quad \bar{\psi}_i = \psi_i^\dagger\gamma^0 = \begin{pmatrix}\chi_i^\dagger & \xi_i^\dagger\end{pmatrix} \ , \tag{71}$$
thus
$$P_L\psi_i = \begin{pmatrix}\chi_i \\ 0\end{pmatrix} \quad \text{and} \quad P_R\psi_i = \begin{pmatrix}0 \\ \xi_i\end{pmatrix} \ . \tag{72}$$
Now we consider the quantity
$$\bar{\psi}_i\gamma^\mu P_L\psi_j = \psi_i^\dagger\gamma^0\gamma^\mu\begin{pmatrix}\chi_j \\ 0\end{pmatrix} = \psi_i^\dagger\gamma^0\begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}\begin{pmatrix}\chi_j \\ 0\end{pmatrix} = \psi_i^\dagger\begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}\begin{pmatrix}0 \\ \bar{\sigma}^\mu\chi_j\end{pmatrix} \tag{73}$$
$$= \begin{pmatrix}\chi_i^\dagger & \xi_i^\dagger\end{pmatrix}\begin{pmatrix}\bar{\sigma}^\mu\chi_j \\ 0\end{pmatrix} = \chi_i^\dagger\bar{\sigma}^\mu\chi_j \ . \tag{74}$$
The Fierz identity for the right-handed portion (ξ_i) of Dirac spinors is given by Peskin Eq. 3.78, by analogy (or Peskin Eq. 3.79) we give the identity of the left handed portions (χ_i):
$$(\bar{\xi}_1\sigma^\mu\xi_2)(\bar{\xi}_3\sigma_\mu\xi_4) = -(\bar{\xi}_1\sigma^\mu\xi_4)(\bar{\xi}_3\sigma_\mu\xi_2) \tag{75}$$
$$(\bar{\chi}_1\bar{\sigma}^\mu\chi_2)(\bar{\chi}_3\bar{\sigma}_\mu\chi_4) = -(\bar{\chi}_1\bar{\sigma}^\mu\chi_4)(\bar{\chi}_3\bar{\sigma}_\mu\chi_2) \ , \tag{76}$$
where $\bar{\chi} = \chi^\dagger$ and $\bar{\xi} = \xi^\dagger$, because they are two-component spinors. Using Equations 74 and 76, we see
$$(\bar{\psi}_1\gamma^\mu P_L\psi_2)(\bar{\psi}_3\gamma^\mu P_L\psi_4) = (\chi_1^\dagger\bar{\sigma}^\mu\chi_2)(\chi_3^\dagger\bar{\sigma}^\mu\chi_4) = -(\bar{\chi}_1\bar{\sigma}^\mu\chi_4)(\bar{\chi}_3\bar{\sigma}_\mu\chi_2) \tag{77}$$
$$= -(\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma^\mu P_L\psi_2) \ . \tag{78}$$

B)

$$(\bar{\psi}_1\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_2)(\bar{\psi}_3\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_4) = -16(\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma^\mu P_L\psi_2)$$

Using the same spinor definition as the previous section, consider terms of the form:

$$\begin{aligned}
\bar{\psi}_i\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_j &= \bar{\psi}_i\gamma^\mu\gamma^\alpha\gamma^\beta \begin{pmatrix}\chi_j\\ 0\end{pmatrix} = \bar{\psi}_i\gamma^\mu\gamma^\alpha \begin{pmatrix} & \sigma^\beta \\ \bar{\sigma}^\beta & \end{pmatrix}\begin{pmatrix}\chi_j\\ 0\end{pmatrix} = \bar{\psi}_i\gamma^\mu \begin{pmatrix} & \sigma^\alpha \\ \bar{\sigma}^\alpha & \end{pmatrix}\begin{pmatrix}0\\ \bar{\sigma}^\beta\chi_j\end{pmatrix} \\
&= \bar{\psi}_i \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}\begin{pmatrix}\sigma^\alpha\bar{\sigma}^\beta\chi_j\\ 0\end{pmatrix} = \psi_i^\dagger\gamma^0 \begin{pmatrix}0\\ \bar{\sigma}^\mu\sigma^\alpha\bar{\sigma}^\beta\chi_j\end{pmatrix} = \begin{pmatrix}\chi_i^\dagger & \xi_i^\dagger\end{pmatrix}\begin{pmatrix}\bar{\sigma}^\mu\sigma^\alpha\bar{\sigma}^\beta\chi_j\\ 0\end{pmatrix} \\
&= \chi_i^\dagger\bar{\sigma}^\mu\sigma^\alpha\bar{\sigma}^\beta\chi_j \ .
\end{aligned}$$

which are eqivalent in form to the terms on the left-hand side of Peskin Eq. 3.82. Using this equation, we see

$$\begin{aligned}
(\bar{\psi}_1\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_2)(\bar{\psi}_3\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_4) &= (\chi_1^\dagger\bar{\sigma}^\mu\sigma^\alpha\bar{\sigma}^\beta\chi_2)(\chi_3^\dagger\bar{\sigma}^\mu\sigma^\alpha\bar{\sigma}^\beta\chi_4) && (79)\\
&= 16(\chi_1^\dagger\bar{\sigma}^\mu\chi_2)(\chi_3^\dagger\bar{\sigma}^\mu\chi_4) && (80)\\
&= 16(\bar{\psi}_1\gamma^\mu P_L\psi_2)(\bar{\psi}_3\gamma^\mu P_L\psi_4) && (81)\\
&= -16(\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma^\mu P_L\psi_2) \ , && (82)
\end{aligned}$$

where between Equations 79 and 80, we have utilized Peskin Eq. 3.82, and at the final step we used Equation 74.

C)

$$\mathrm{Tr}[\Gamma^M\Gamma^N] = 4\delta^{MN}, \text{ with } \Gamma^M \in \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma_5\gamma^\mu, \gamma_5\}$$

It is clear to see the scalar Γ matrix is properly normalized:

$$\mathrm{Tr}[\mathbb{1}\cdot\mathbb{1}] = \mathrm{Tr}[\mathbb{1}] = 4 \ . \tag{83}$$

We will check the vector components:

$$\mathrm{Tr}[a\gamma^0 a\gamma^0] = a^2\,\mathrm{Tr}[(\gamma^0)^2] = a^2\,\mathrm{Tr}[\mathbb{1}] = 4a^2 \ , \tag{84}$$

equating this with $4\delta^{00}$, we see $a = 1$. For space-like components, we have

$$\mathrm{Tr}[a\gamma^i b\gamma^j] = ab\,\mathrm{Tr}[\gamma^i\gamma^j] = 4abg^{ij} = -4ab\delta^{ij} \ , \tag{85}$$

using Equation 43 - equating this with $4\delta^{ij}$ we see this is satisfied for $a = b = i$. Let us check the cross terms

$$\mathrm{Tr}[\gamma^0 i\gamma^j] = ig^{0i} = 0 \ , \tag{86}$$

which also satisfies equating with $4\delta^{\mu\nu}$. Now we will check the tensorial components of Γ, first noting that

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \ . \tag{87}$$

Taking the trace of the square of $\sigma^{\mu\nu}$ yields

$$\begin{aligned}
\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\mu\nu}] &= -\frac{1}{4}\,\mathrm{Tr}\,\{[\gamma^\mu,\gamma^\nu][\gamma^\mu,\gamma^\nu]\} && (88)\\
&= -\frac{1}{4}\,\mathrm{Tr}\,\{\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\mu\gamma^\nu\gamma^\mu\} \ , && (89)
\end{aligned}$$

we note that if $\mu = \nu$ the right-hand side vanishes, so using the anticommutation relation, this is

$$\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\mu\nu}] = -\frac{1}{4}\,\mathrm{Tr}\left\{\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu - (-1)\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu - (-1)\gamma^\mu\gamma^\nu\gamma^\nu\gamma^\mu + (-1)^2\gamma^\nu\gamma^\mu\gamma^\nu\gamma^\mu\right\} \tag{90}$$

$$= -\frac{1}{4}\,\mathrm{Tr}\left\{4\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu\right\} = -\,\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu]\ , \tag{91}$$

using Equation 46 this is

$$\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\mu\nu}] = -4(g^{\mu\nu}g^{\mu\nu} - g^{\mu\mu}g^{\nu\nu} + g^{\mu\nu}g^{\nu\mu})\ , \tag{92}$$

since we have previously stated $\mu \neq \nu$ this simplifies to

$$\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\mu\nu}] = -4g^{\mu\mu}g^{\nu\nu}\ , \tag{93}$$

so

$$\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\mu\nu}] = \begin{cases} -4 & \text{for} \quad \mu,\nu \text{ both space-like} \\ +4 & \text{for} \quad \mu,\nu \text{ one space-like and one time-like} \end{cases}\ , \tag{94}$$

noting that the case for both indeces being time-like vanishes. Given that $\mu \neq \nu$, we have $\sigma^{\mu\nu} \in \{\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{13}, \sigma^{23}\}$. Consider the case for for one space-like and one-time like Lorentz index:

$$\mathrm{Tr}[\Gamma^A\Gamma^A] = a^2\,\mathrm{Tr}[\sigma^{0j}\sigma^{0j}] = +4a^2\ , \tag{95}$$

equating this with $4\delta^{AA}$, we see $a = 1$. In the case for two space-like indeces, we have

$$\mathrm{Tr}[\Gamma^A\Gamma^A] = a^2\,\mathrm{Tr}[\sigma^{jk}\sigma^{jk}] = -4a^2\ , \tag{96}$$

in which case $a = i$. For the pseudoscalar case it is easy to see

$$\mathrm{Tr}[a\gamma_5 a\gamma_5] = a^2\,\mathrm{Tr}[(\gamma_5)^2] = a^2\,\mathrm{Tr}[\mathbb{1}] = 4a^2\ , \tag{97}$$

so we have $a = 1$ in the desired basis. The pseudovector entries are

$$\mathrm{Tr}[a\gamma_5\gamma^\mu a\gamma_5\gamma^\mu] = a^2\,\mathrm{Tr}[\gamma_5\gamma^\mu\gamma_5\gamma^\mu] = -a^2\,\mathrm{Tr}[\gamma^\mu\gamma_5\gamma_5\gamma^\mu] = -a^2\,\mathrm{Tr}[\gamma^\mu\mathbb{1}\gamma^\mu] \tag{98}$$

$$= -a^2\,\mathrm{Tr}[(\gamma^\mu)^2]\ , \tag{99}$$

if $\mu = 0$, we have

$$\mathrm{Tr}[a\gamma_5\gamma^0 a\gamma_5\gamma^0] = -a^2\,\mathrm{Tr}[(\gamma^0)^2] = -a^2\,\mathrm{Tr}[\mathbb{1}] = -4a^2\ , \tag{100}$$

and if $\mu = j$, we have

$$\mathrm{Tr}[a\gamma_5\gamma^j a\gamma_5\gamma^j] = -a^2\,\mathrm{Tr}[(\gamma^j)^2] = a^2\,\mathrm{Tr}[\mathbb{1}] = 4a^2\ , \tag{101}$$

which we equate with 4 in the desired basis. For $\mu = 0$, $a = i$, and for $\mu = j$, $a = 1$. Thus the properly normalized basis for Γ^A is

$$\Gamma^A \in \{\mathbb{1}, \gamma^0, i\gamma^j, \sigma^{0j}, i\sigma^{jk}, \gamma_5, \gamma_5\gamma^0, i\gamma_5\gamma^j\}\ , \tag{102}$$

for $j \neq k$, and in this basis

$$\mathrm{Tr}[\Gamma^A\Gamma^B] = 4\delta^{AB}\ . \tag{103}$$

D)

$$(\bar{\psi}_1\Gamma^M\psi_2)(\bar{\psi}_3\Gamma^N\psi_4) = \sum_{PQ}\frac{1}{16}\,\mathrm{Tr}[\Gamma^P\Gamma^M\Gamma^Q\Gamma^N](\bar{\psi}_1\Gamma^P\psi_4)(\bar{\psi}_3\Gamma^Q\psi_2)$$

The quantity $\bar{\psi}_i\Gamma^A\psi_j$ is a number, and the trace of a number is itself, so

$$\bar{\psi}_i\Gamma^A\psi_j = \mathrm{Tr}[\bar{\psi}_i\Gamma^A\psi_j] = \mathrm{Tr}[\psi_j\bar{\psi}_i\Gamma^A]\ , \tag{104}$$

and we note that $\psi_j\bar{\psi}_i$ is a 2×2 matrix (of 2×2 matrices). The completeness relation from quantum mechanics is $\mathbb{1} = \sum_k |k\rangle\langle k|$, by analogy, we have

$$\mathbb{1} = \sum_A (\Gamma^A)^\dagger\Gamma^A\ , \tag{105}$$

but since $\{\Gamma^A\}$ are Hermetian, this becomes

$$\mathbb{1} = \sum_A \Gamma^A\Gamma^A\ . \tag{106}$$

We can then expand any matrix in this basis: take, for instance, $\psi_j\bar{\psi}_i$:

$$\psi_j\bar{\psi}_i = \sum_A x^A_{ji}\Gamma^A\ , \tag{107}$$

where x^A_{ji} is a number. Inserting this into Equation 104, we have

$$\bar{\psi}_i\Gamma^A\psi_j = \mathrm{Tr}[\bar{\psi}_i\Gamma^A\psi_j] = \mathrm{Tr}[\sum_Z x^Z_{ji}\Gamma^Z\Gamma^A] = \sum_Z x^Z_{ji}\,\mathrm{Tr}[\Gamma^Z\Gamma^A] = \sum_Z 4x^Z_{ji}\delta^{ZA}\ . \tag{108}$$

We can write down the general form for expressing a quantity in a different basis:

$$(\bar{\psi}_1\Gamma^A\psi_2)(\bar{\psi}_3\Gamma^B\psi_4) = \sum_{C,D} C^{AB}_{CD}(\bar{\psi}_1\Gamma^C\psi_4)(\bar{\psi}_3\Gamma^D\psi_2)\ , \tag{109}$$

where C^{AB}_{CD} is some undetermined coefficient. Consider the right-hand side:

$$\sum_{C,D} C^{AB}_{CD}(\bar{\psi}_1\Gamma^C\psi_4)(\bar{\psi}_3\Gamma^D\psi_2) = \sum_{C,D} C^{AB}_{CD}\left\{\sum_Z 4x^Z_{41}\delta^{ZC}\right\}\left\{\sum_Y 4x^Y_{23}\delta^{YD}\right\} \tag{110}$$

$$= \sum_{C,D} C^{AB}_{CD}\left\{16x^C_{41}x^D_{23}\right\}\ , \tag{111}$$

and the left-hand side:

$$(\bar{\psi}_1\Gamma^A\psi_2)(\bar{\psi}_3\Gamma^B\psi_4) = \mathrm{Tr}[\bar{\psi}_1\Gamma^A\psi_2\bar{\psi}_3\Gamma^B\psi_4] = \mathrm{Tr}[\psi_4\bar{\psi}_1\Gamma^A\psi_2\bar{\psi}_3\Gamma^B]\ , \tag{112}$$

since both quantities in parenthesis are just numbers. Using Equation 107, this is equivalent to

$$(\bar{\psi}_1\Gamma^A\psi_2)(\bar{\psi}_3\Gamma^B\psi_4) = \mathrm{Tr}[\sum_Z x^Z_{41}\Gamma^Z\Gamma^A\sum_Y x^Y_{23}\Gamma^Y\Gamma^B] = \sum_{Z,Y}\mathrm{Tr}[\Gamma^Z\Gamma^A\Gamma^Y\Gamma^B]x^Z_{41}x^Y_{23}\ , \tag{113}$$

if we relabel the dummy indeces: $Z \to C$ and $Y \to D$, this is

$$(\bar{\psi}_1 \Gamma^A \psi_2)(\bar{\psi}_3 \Gamma^B \psi_4) = \sum_{C,D} \mathrm{Tr}[\Gamma^C \Gamma^A \Gamma^D \Gamma^B] x_{41}^C x_{23}^D \ . \tag{114}$$

Comparing this result to Equation 111, we see

$$C_{CD}^{AB} = \frac{1}{16} \mathrm{Tr}[\Gamma^C \Gamma^A \Gamma^D \Gamma^B] \ , \tag{115}$$

proving

$$(\bar{\psi}_1 \Gamma^M \psi_2)(\bar{\psi}_3 \Gamma^N \psi_4) = \sum_{PQ} \frac{1}{16} \mathrm{Tr}[\Gamma^P \Gamma^M \Gamma^Q \Gamma^N](\bar{\psi}_1 \Gamma^P \psi_4)(\bar{\psi}_3 \Gamma^Q \psi_2) \ . \tag{116}$$