Dylan J. Temples: Solution Set Six

Quantum Field Theory I
Quantum Field Theory and the Standard Model - M. Schwartz
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1 Scalar Yukawa theory.

Consider the scalar Yukawa theory we discussed in class

$$\mathcal{L} = \int \mathrm{d}^4 x \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) - \frac{1}{2} m_\phi^2 \phi^2 + \partial_\mu \psi^\dagger \partial^\mu \psi - m_\psi^2 \psi^\dagger \psi - g\psi^\dagger \psi \phi \ . \tag{1}$$

In the interaction picture, the quantum field is written in the same way as the free field:

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_k}} (a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}), \quad \psi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_k}} (b_k e^{-ik\cdot x} + c_k^\dagger e^{ik\cdot x}) \ . \tag{2}$$

Furthermore, define the amplitude $\mathcal{A}$ of a particular process as

$$\langle f | S - 1 | i \rangle = i\mathcal{A}\delta^{(4)} \left(\sum_i k_i - \sum_f k_f \right) \ . \tag{3}$$

A) Derive the following Wick contractions for the complex scalar ψ:

$$\overbracket{\psi^\dagger(x)\psi^\dagger}(y) = 0, \quad \overbracket{\psi(x)\psi}(y) = 0, \quad \overbracket{\psi^\dagger(x)\psi}(y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m_\psi^2 + i\epsilon} e^{-ik\cdot(x-y)} \ . \tag{4}$$

The Wick contraction of two fields is defined as

$$\overbracket{AB} = \mathcal{T}\{AB\} - :AB: \ , \tag{5}$$

where $\mathcal{T}$ is the time-ordered product operator and $:x:$ denotes the "normal ordering" with annihilation operators to the right of creation operators. Also let us note the form of the $\psi^\dagger$ field:

$$\psi^\dagger(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_k}} (b_k^\dagger e^{ik\cdot x} + c_k e^{ik\cdot x}) \ . \tag{6}$$

Let us define the following field-operator terms:

$$\psi_-(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_k}} b_k e^{-ik\cdot x} \qquad \psi_+(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_k}} c_k^\dagger e^{ik\cdot x} \tag{7}$$

$$\psi_-^\dagger(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_k}} b_k^\dagger e^{ik\cdot x} \qquad \psi_+^\dagger(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_k}} c_k e^{ik\cdot x} \ . \tag{8}$$

Let us assume that $x^0 > y_0$, so that

$$\mathcal{T}\left\{\psi(x)\psi(y)\right\} = \psi_-(x)\psi_-(y) + \psi_+(x)\psi_-(y) + \psi_-(x)\psi_+(y) + \psi_+(x)\psi_+(y) \ , \tag{9}$$

but we note that this is almost normal ordered:

$$:\psi(x)\psi(y) := \psi_-(x)\psi_-(y) + \psi_+(x)\psi_-(y) + \psi_+(y)\psi_-(x) + \psi_+(x)\psi_+(y) \ . \tag{10}$$

The term $\psi_-(x)\psi_+(y)$ has an operator structure $b_k c^\dagger_{k'}$, but these commute, so this term is equivalent to the normal ordering, and thus

$$\mathcal{T}\left\{\psi(x)\psi(y)\right\} =: \psi(x)\psi(y) : \ , \tag{11}$$

so from the definition of the Wick contraction (Equation 5), we have

$$\overbracket{\psi(x)\psi}(y) = 0, \tag{12}$$

this reasoning holds exactly for $y^0 > x^0$ as well. Still assuming $x^0 > y_0$, consider:

$$\mathcal{T}\left\{\psi^\dagger(x)\psi^\dagger(y)\right\} = \psi^\dagger_-(x)\psi^\dagger_-(y) + \psi^\dagger_+(x)\psi^\dagger_-(y) + \psi^\dagger_-(x)\psi^\dagger_+(y) + \psi^\dagger_+(x)\psi^\dagger_+(y) \ , \tag{13}$$

but again we note this is almost normal ordered:

$$:\psi^\dagger(x)\psi^\dagger(y) := \psi^\dagger_-(x)\psi^\dagger_-(y) + \psi^\dagger_-(y)\psi^\dagger_+(x) + \psi^\dagger_-(x)\psi^\dagger_+(y) + \psi^\dagger_+(x)\psi^\dagger_+(y) \ , \tag{14}$$

because the $\psi^\dagger_+$ has an annihilation operator while the $\psi^\dagger_-$ has a creation operator. The non-normal ordered term in the time-ordered product has an operator structure $c_k b^\dagger_{k'}$, which commute, so this term is equivalent to its normal-ordering. Thus,

$$\mathcal{T}\left\{\psi^\dagger(x)\psi^\dagger(y)\right\} =: \psi^\dagger(x)\psi^\dagger(y) : \ , \tag{15}$$

and from the definition of Wick contraction, we have

$$\overbracket{\psi^\dagger(x)\psi^\dagger}(y) = 0, \tag{16}$$

and again, this reasoning holds for $y^0 > x^0$. Finally, we investigate (with $x^0 > y^0$):

$$\mathcal{T}\left\{\psi^\dagger(x)\psi(y)\right\} = \psi^\dagger_-\psi_- + \psi^\dagger_+\psi_- + \psi^\dagger_-\psi_+ + \psi^\dagger_+\psi_+ \ , \tag{17}$$

which has operator structure:

$$b^\dagger_k b_{k'} + c_k b_{k'} + b^\dagger_k c_{k'} + c_k c^\dagger_{k'} \ , \tag{18}$$

which is normal ordered except for the last term. This operator structure is the same as

$$b^\dagger_k b_{k'} + c_k b_{k'} + b^\dagger_k c_{k'} + c^\dagger_{k'} c_k + [c_k, c^\dagger_{k'}] \ , \tag{19}$$

which is normal ordered. Therefore

$$\overbracket{\psi^\dagger(x)\psi}(y) = \mathcal{T}\left\{\psi^\dagger(x)\psi(y)\right\} - :\psi^\dagger(x)\psi(y): \tag{20}$$

$$= \int \frac{\mathrm{d}^3k e^{-ik\cdot x}}{(2\pi)^3\sqrt{2\omega_k}} \int \frac{\mathrm{d}^3k' e^{ik'\cdot y}}{(2\pi)^3\sqrt{2\omega_k'}}[c_k, c_{k'}^\dagger] \tag{21}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_k}} \int \frac{\mathrm{d}^3k'}{(2\pi)^3\sqrt{2\omega_k'}}\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{-ik\cdot(x-y)} \tag{22}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)} \ , \tag{23}$$

but this is exactly the form of the Feynman propagator for a real scalar field:

$$D_F(x-y) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)} \ , \tag{24}$$

so we conclude that

$$\overbracket{\psi^\dagger(x)\psi}(y) = D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} = D_F(x-y) = \overbracket{\phi(x)\phi}(y) \ , \tag{25}$$

as expected.

B) Compute $i\mathcal{A}$ in the centre-of-mass frame for the scattering process $\psi(k_1) + \psi^\dagger(k_2) \to \psi(p_1) + \psi^\dagger(p_2)$ using Dyson's formula. (Do not use any Feynman rules here!)

The operator S is defined to be

$$S = \mathcal{T}\left\{ e^{-i\int_{-\infty}^{\infty} H_I(t)} \right\} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \mathcal{T}\left\{ (-i)^\ell \left(\int_{-\infty}^{\infty} H_I(t) \right)^\ell \right\} , \tag{26}$$

where $H_I(t) = \int \mathrm{d}^3x g\psi^\dagger\psi\phi$. Then the matrix element $\langle f|S-1|i\rangle$ can be written

$$\langle f| \sum_{\ell=1}^{\infty} \frac{(-ig)^\ell}{\ell!} \mathcal{T}\left\{ \left(\int \mathrm{d}^4x \psi^\dagger(x)\psi(x)\phi(x) \right)^\ell \right\} |i\rangle . \tag{27}$$

If we look at the scattering process, we see it must be propagated by the real scalar ϕ and so we need terms with one creation and one annihilation operator for this field, and is thus a second order process. From this we conclude the $\ell = 1$ channel does not contribute. The leading order term is then

$$\langle f| \frac{(-ig)^2}{2} \mathcal{T}\left\{ \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2) \right\} |i\rangle . \tag{28}$$

We will drop the arguments for the notation $\psi(x_i) \to \psi_i$, and similarly for $\psi^\dagger$ and ϕ. The final and initial states are

$$|i\rangle = |k_1k_2\rangle \quad \text{and} \quad |f\rangle = |p_1p_2\rangle , \tag{29}$$

so the matrix element can be written

$$\frac{(-ig)^2}{2} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \langle p_1p_2| \mathcal{T}\left\{ \psi_1^\dagger\psi_1\phi_1\psi_2^\dagger\psi_2\phi_2 \right\} |k_1k_2\rangle . \tag{30}$$

We can write out the time-ordered product:

$$\mathcal{T}\left\{ \psi_1^\dagger\psi_1\phi_1\psi_2^\dagger\psi_2\phi_2 \right\} =: \psi_1^\dagger\psi_1\phi_1\psi_2^\dagger\psi_2\phi_2 : + \overbracket{\phi_1\phi_2} : \psi_1^\dagger\psi_1\psi_2^\dagger\psi_2 : + \dots , \tag{31}$$

were the ... represents all other permutations with two fields contracted. In order to have final and initial states with no real scalars, we must contract the fields ϕ_1 and ϕ_2 (all other contraction permutations result in vanishing terms):

$$\frac{(-ig)^2}{2} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \overbracket{\phi_1\phi_2} \langle \psi(p_1)\psi^\dagger(p_2)| : \psi_1^\dagger\psi_1\psi_2^\dagger\psi_2 : |\psi(k_1)\psi^\dagger(k_2)\rangle . \tag{32}$$

Note that the term with the normal ordering of all six fields vanishes. This is because the right-most operator will be a_k (from the real scalar field) - since the asymptotic states do not have any ϕ particles, the state is a vacuum relative to the field ϕ and acting the annihilation operator on the vacuum yields zero. In general, this term will vanish unless the number of asymptotic states is the same as the number of fields. In the scalar Yukawa interaction this is never the case, so in the rest of the problems we will use this fact. We can define the contraction of a complex scalar field with an single-particle external state as

$$\overbracket{\psi(x_1)\,|k_1}\rangle = e^{-ix_1\cdot k_1} |0\rangle \tag{33}$$

$$\langle \overbracket{p_1|\,\psi}^\dagger(x_1) = \langle 0| e^{ix_1\cdot p_1} , \tag{34}$$

and the action of the complex scalar field on a single-antiparticle state:

$$\psi^\dagger(x_1)\,|k_1^\dagger\rangle = e^{-ix_1\cdot k_1}\,|0\rangle \tag{35}$$

$$\langle p_1^\dagger|\,\psi(x_1) = \langle 0|\,e^{ix_1\cdot p_1}\ , \tag{36}$$

where we've used the notation $|\psi^\dagger(k)\rangle = |k^\dagger\rangle$. Let's investigate the cross-terms:

$$\psi(x_1)\,|k_1^\dagger\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_k}}(b_k e^{-ik\cdot x} + c_k^\dagger e^{ik\cdot x})c_{k_1}^\dagger\,|0\rangle\ , \tag{37}$$

the second term does not contribute, but the first looks like

$$b_k c_{k_1}^\dagger\,|0\rangle = [b_k, c_{k_1}^\dagger]\,|0\rangle - c_{k_1}^\dagger b_k\,|0\rangle\ , \tag{38}$$

since the commutator is zero, and the annihilation of the vacuum is zero, this vanishes, as well as its adjoint. Using this logic, we conclude that

$$\psi^\dagger(x_1)\,|k_1\rangle = 0 \tag{39}$$

as well. From Equation 32 we have

$$\frac{(-ig)^2}{2}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\phi_1\phi_2\,\langle\psi(p_1)\psi^\dagger(p_2)|:\psi_1^\dagger\psi_1\psi_2^\dagger\psi_2:|\psi(k_1)\psi^\dagger(k_2)\rangle\ . \tag{40}$$

Now consider just the matrix element, which has four ways of being contracted (*i.e.*, each field with each external particle). However two of these are redundant under interchange of of the indeces 1 and 2, this kills the factor of 1/2 acquired from expanding the exponential. The first pair of redundant contractions is

$$\langle p_1p_2^\dagger|:\psi_1^\dagger\psi_1\psi_2^\dagger\psi_2:|k_1k_2^\dagger\rangle \tag{41}$$

$$\langle p_1p_2^\dagger|:\psi_1^\dagger\psi_1\psi_2^\dagger\psi_2:|k_1k_2^\dagger\rangle\ , \tag{42}$$

which equal

$$e^{-i(x_2\cdot k_1)}e^{-i(x_2\cdot k_2)}e^{+i(x_1\cdot p_2)}e^{+i(x_1\cdot p_1)} = e^{-ix_1\cdot(-p_2-p_1)}e^{-ix_2\cdot(k_1+k_2)}\ . \tag{43}$$

The second pair of redundant contractions is

$$\langle p_1p_2^\dagger|:\psi_1^\dagger\psi_1\psi_2^\dagger\psi_2:|k_1k_2^\dagger\rangle \tag{44}$$

$$\langle p_1p_2^\dagger|:\psi_1^\dagger\psi_1\psi_2^\dagger\psi_2:|k_1k_2^\dagger\rangle\ , \tag{45}$$

which equal

$$e^{-i(x_2\cdot k_1)}e^{-i(x_1\cdot k_2)}e^{+i(x_2\cdot p_1)}e^{+i(x_1\cdot p_2)} = e^{-ix_1\cdot(k_2-p_2)}e^{-ix_2\cdot(k_1-p_1)}\ . \tag{46}$$

Collecting our results, and inserting the propagator for the contraction of two real scalars:

$$\phi_1(x_1)\phi_2(x_2) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\phi^2 + i\epsilon}e^{-ip\cdot(x_1-x_2)} \ , \tag{47}$$

we can write the matrix element:

$$\langle\psi(p_1)\psi^\dagger(p_2)|S-1|\psi(k_1)\psi^\dagger(k_2)\rangle =$$
$$(-ig)^2\int d^4x_1 d^4x_2 \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x_1-x_2)}}{p^2 - m_\phi^2 + i\epsilon}\left\{e^{-ix_1\cdot(-p_2-p_1)}e^{-ix_2\cdot(k_1+k_2)} + e^{-ix_1\cdot(k_2-p_2)}e^{-ix_2\cdot(k_1-p_1)}\right\}.$$

Here we can write the exponential contribution of the propagator as

$$e^{-ip\cdot(x_1-x_2)} = e^{-ix_1\cdot(p)}e^{-ix_2\cdot(-p)} \ . \tag{48}$$

If we combine exponentials and swap the orders of integration, we have, for the position-space integrals:

$$\int d^4x_1 e^{-ix_1\cdot(-p_2-p_1+p)}\int d^4x_2 e^{-ix_2\cdot(k_1+k_2-p)} = \delta^{(4)}(p-[p_1+p_2])\delta^{(4)}([k_1+k_2]-p) \tag{49}$$
$$\int d^4x_1 e^{-ix_1\cdot(k_2-p_2+p)}\int d^4x_2 e^{-ix_2\cdot(k_1-p_1-p)} = \delta^{(4)}(p-[p_2-k_2])\delta^{(4)}([k_1-p_1]-p) \ . \tag{50}$$

We can now integrate over the momentum space, using the second delta function in each term to eliminate p:

$$i(-ig)^2\left[\frac{\delta^{(4)}([k_1+k_2]-[p_1+p_2])}{[k_1+k_2]^2 - m_\phi^2 + i\epsilon} + \frac{\delta^{(4)}([k_1-p_1]-[p_2-k_2])}{[k_1-p_1]^2 - m_\phi^2 + i\epsilon}\right] \ , \tag{51}$$

which is equivalent to using the first delta function in each term to eliminate p:

$$i(-ig)^2\left[\frac{\delta^{(4)}([k_1+k_2]-[p_1+p_2])}{[p_1+p_2]^2 - m_\phi^2 + i\epsilon} + \frac{\delta^{(4)}([k_1-p_1]-[p_2-k_2])}{[p_2-k_2]^2 - m_\phi^2 + i\epsilon}\right] \ . \tag{52}$$

Using the first expression, we can factor out the delta function yielding

$$i(-ig)^2\left[\frac{1}{(k_1+k_2)^2 - m_\phi^2 + i\epsilon} + \frac{1}{(k_1-p_1)^2 - m_\phi^2 + i\epsilon}\right]\delta^{(4)}([k_1+k_2]-[p_1+p_2]) \ , \tag{53}$$

and from the definition of the amplitude, we have

$$\mathcal{A} = (-ig)^2\left[\frac{1}{(k_1+k_2)^2 - m_\phi^2 + i\epsilon} + \frac{1}{(k_1-p_1)^2 - m_\phi^2 + i\epsilon}\right] \tag{54}$$
$$= (-ig)^2\left[\frac{1}{(p_1+p_2)^2 - m_\phi^2 + i\epsilon} + \frac{1}{(p_2-k_2)^2 - m_\phi^2 + i\epsilon}\right] \ . \tag{55}$$

C) Draw the relevant Feynman diagram(s) for the process in (b). Then use Feynman rules to re-derive your answer in (b).

The Feynman diagrams that describe the scattering process $\psi(k_1) + \psi^\dagger(k_2) \to \psi(p_1) + \psi^\dagger(p_2)$ are given in Figure 1. The process in the left diagram (s-channel) is propagated by a real scalar with momentum $k_1 + k_2 = p_1 + p_2$ (found by conserving momentum at each vertex). It is a second order process (so we obtain a factor of $-ig$ for each vertex), yielding the amplitude:

$$i\mathcal{A}_t = (-ig)^2 \frac{i}{(k_1 + k_2)^2 - m_\phi^2 + i\epsilon} \ . \tag{56}$$

Similarly, for the process in the right diagram(t-channel) is propagated by a real scalar of momentum $k_1 - p_1 = p_2 - k_2$, yielding the amplitude:

$$i\mathcal{A}_t = (-ig)^2 \frac{i}{(k_1 - p_1)^2 - m_\phi^2 + i\epsilon} \ . \tag{57}$$

Note the u-channel process is not allowed because it cannot conserve $U(1)$ charge, see Figure 2. These results are consistent with the amplitudes found using Dyson's method.

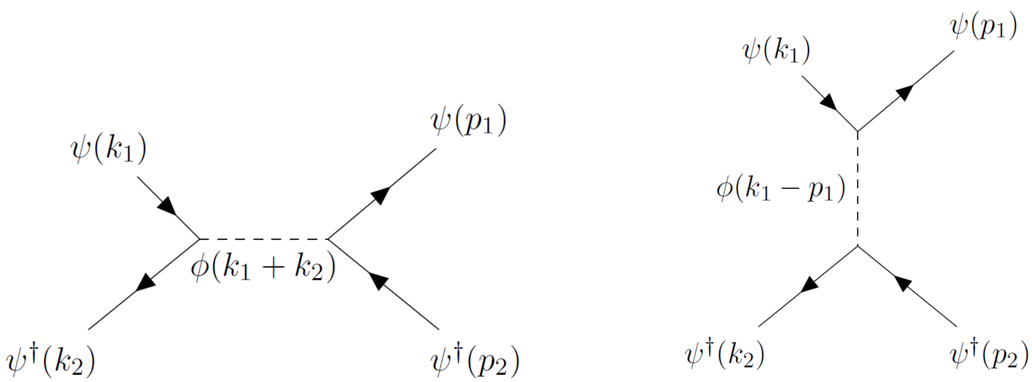

Figure 1: Feynman diagrams corresponding to the process $\psi(k_1) + \psi^\dagger(k_2) \to \psi(p_1) + \psi^\dagger(p_2)$.

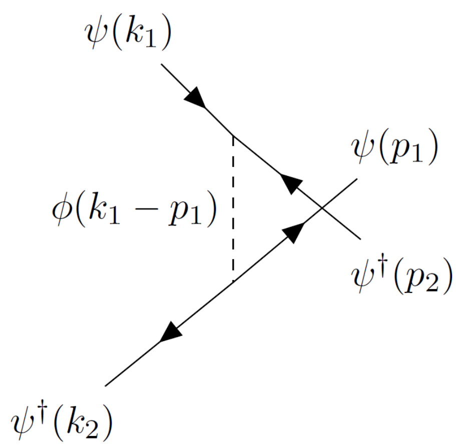

Figure 2: Forbidden u-channel process for the scattering process in part C, note $U(1)$ charge is not conserved.

D) Compute $i\mathcal{A}$ in the centre-of-mass frame for the scattering process $\psi(k_1) + \phi(k_2) \to \psi(p_1) + \phi(p_2)$ using Dyson's formula. (Do not use any Feynman rules here!)

The matrix element for this process is given by

$$\langle\psi(p_1)\phi(p_2)|\frac{(-ig)^2}{2!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathcal{T}\left\{\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\right\}|\psi(k_1)\phi(k_2)\rangle \tag{58}$$

so let us consider the time-ordered product:

$$\mathcal{T}\left\{\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\right\} =$$
$$: \psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2) : + \overbracket{\psi^\dagger(x_1)\psi^\dagger}(x_2) : \psi(x_1)\phi(x_1)\psi(x_2)\phi(x_2) : + \dots ,$$

again where the ... represents the remaining permutations of contractions of two fields. We know we must contract only two fields because there is a total of six fields in the operator, but only for asymptotic states. We cannot contract the real scalar fields because we would be unable to create/annihilate the asymptotic real scalar states. Furthermore, from part (a) we know the only nonzero contractions of complex scalar fields have one ψ field and one $\psi^\dagger$ field. The four non-vanishing terms with two fields contracted are

$$\overbracket{\psi_1^\dagger\psi_1} :\phi_1\psi_2^\dagger\psi_2\phi_2 : \tag{59}$$
$$\overbracket{\psi_2^\dagger\psi_2} :\psi_1^\dagger\psi_1\phi_1\phi_2 : \tag{60}$$
$$\overbracket{\psi_1^\dagger\psi_2} :\psi_1\phi_1\psi_2^\dagger\phi_2 : \tag{61}$$
$$\overbracket{\psi_1\psi_2^\dagger} :\psi_1^\dagger\phi_1\psi_2\phi_2 : , \tag{62}$$

where we have dropped the position argument of the fields for a subscript. Here it is useful to note

$$\overbracket{\psi^\dagger(x)\psi}(y) = \overbracket{\psi(y)\psi^\dagger}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m_\psi^2 + i\epsilon}e^{-ik\cdot(x-y)} , \tag{63}$$

but in this case:

$$\overbracket{\psi_1^\dagger\psi_1} = \overbracket{\psi_2^\dagger\psi_2} = 0 , \tag{64}$$

so we are left with the terms:

$$\overbracket{\psi_1^\dagger\psi_2} :\psi_1\phi_1\psi_2^\dagger\phi_2 : \tag{65}$$
$$\overbracket{\psi_1\psi_2^\dagger} :\psi_1^\dagger\phi_1\psi_2\phi_2 : , \tag{66}$$

but we note (from Equation 63) these are redundant under interchange of the indeces. This eliminates the factor of two obtained from expanding the exponential. We are then left with the matrix element:

$$(-ig)^2\int \mathrm{d}^4x_1\mathrm{d}^4x_2\overbracket{\psi_1^\dagger\psi_2}\langle p_1p_2| :\psi_1\phi_1\psi_2^\dagger\phi_2 : |k_1k_2\rangle , \tag{67}$$

where we can insert the propagator and interchange the order of integration to obtain:

$$(-ig)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \frac{i e^{ik\cdot(x_1 - x_2)}}{k^2 - m_\psi^2 + i\epsilon} \langle p_1 p_2 | : \psi_1 \phi_1 \psi_2^\dagger \phi_2 : | k_1 k_2 \rangle \ . \tag{68}$$

Note we had to be careful with the sign of the exponential in the propagator due to the fact we switched the position-space coordinates to make the two normal ordered terms equal. There is only one way to contract the complex scalar fields with external particles:

$$\langle p_1 p_2 | : \psi_1 \phi_1 \psi_2^\dagger \phi_2 : | k_1 k_2 \rangle = e^{-ix_1\cdot k_1} e^{+ix_2\cdot p_1} \langle 0|0\rangle \otimes \langle p_2 | : \phi_1 \phi_2 : | k_2 \rangle \ . \tag{69}$$

There are two ways to contract the real scalar fields:

$$\langle p_2 | : \phi_1 \phi_2 : | k_2 \rangle = e^{-ix_2\cdot k_2} e^{+ix_1\cdot p_2} \langle 0|0\rangle \tag{70}$$

$$\langle p_2 | : \phi_1 \phi_2 : | k_2 \rangle = e^{-ix_1\cdot k_2} e^{+ix_2\cdot p_2} \langle 0|0\rangle \ . \tag{71}$$

Combining these results, we have that the matrix element is

$$(-ig)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \frac{i e^{ip\cdot(x_1 - x_2)}}{p^2 - m_\psi^2 + i\epsilon} e^{-ix_1\cdot k_1} e^{+ix_2\cdot p_1} \left\{ e^{-ix_2\cdot k_2} e^{+ix_1\cdot p_2} + e^{-ix_1\cdot k_2} e^{+ix_2\cdot p_2} \right\}$$

the exponential factors for the first term simplify to

$$e^{-ix_1\cdot(-p)} e^{-ix_2\cdot(p)} e^{-ix_1\cdot k_1} e^{-ix_2\cdot(-p_1)} e^{-ix_2\cdot k_2} e^{-ix_1\cdot(-p_2)} = e^{-ix_1\cdot(-p+k_1-p_2)} e^{-ix_2\cdot(p-p_1+k_2)} \ , \tag{72}$$

and in the second term

$$e^{-ix_1\cdot(-p)} e^{-ix_2\cdot(p)} e^{-ix_1\cdot k_1} e^{-ix_2\cdot(-p_1)} e^{-ix_1\cdot k_2} e^{-ix_2\cdot(-p_2)} = e^{-ix_1\cdot(-p+k_1+k_2)} e^{-ix_2\cdot(p-p_1-p_2)} \ . \tag{73}$$

We can then perform the integration over the position-space coordinates x_1 and x_2 for both terms yielding

$$\int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \to \delta^{(4)}([k_1 - p_2] - p) \delta^{(4)}(p - [p_1 - k_2]) \tag{74}$$

$$\int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \to \delta^{(4)}([k_1 + k_2] - p) \delta^{(4)}(p - [p_1 + p_2]) \ , \tag{75}$$

for the first and second terms, respectively. What remains in calculating the matrix element is a momentum-space integral:

$$i(-ig)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left\{ \frac{\delta^{(4)}([k_1 - p_2] - p)\delta^{(4)}(p - [p_1 - k_2])}{p^2 - m_\psi^2 + i\epsilon} + \frac{\delta^{(4)}([k_1 + k_2] - p)\delta^{(4)}(p - [p_1 + p_2])}{p^2 - m_\psi^2 + i\epsilon} \right\} \ ,$$

selecting one of the delta functions to integrate over yields two equivalent expressions (depending on the selection of the delta function. Integrating out the second delta yields

$$i(-ig)^2 \left\{ \frac{\delta^{(4)}([k_1 - p_2] - [p_1 - k_2])}{[p_1 - k_2]^2 - m_\psi^2 + i\epsilon} + \frac{\delta^{(4)}([k_1 + k_2] - [p_1 + p_2])}{[p_1 + p_2]^2 - m_\psi^2 + i\epsilon} \right\} \ , \tag{76}$$

while integrating out the first yields

$$i(-ig)^2 \left\{ \frac{\delta^{(4)}([k_1 - p_2] - [p_1 - k_2])}{[k_1 - p_2]^2 - m_\psi^2 + i\epsilon} + \frac{\delta^{(4)}([k_1 + k_2] - [p_1 + p_2])}{[k_1 + k_2]^2 - m_\psi^2 + i\epsilon} \right\} , \tag{77}$$

note that each delta function has the same argument which enforces overall conservation of momentum. Using the definition of the process amplitude, we get

$$\mathcal{A} = (-ig)^2 \left\{ \frac{1}{(p_1 - k_2)^2 - m_\psi^2 + i\epsilon} + \frac{1}{(p_1 + p_2)^2 - m_\psi^2 + i\epsilon} \right\} \tag{78}$$

$$= (-ig)^2 \left\{ \frac{1}{(k_1 - p_2)^2 - m_\psi^2 + i\epsilon} + \frac{1}{(k_1 + k_2)^2 - m_\psi^2 + i\epsilon} \right\} . \tag{79}$$

E) Draw the relevant Feynman diagram(s) for the process in (d). Then use Feynman rules to re-derive your answer in (d).

The s-channel process is shown in the left figure of Figure 3. Conserving momentum at each vertex yields:

$$k_1 + k_2 = p_\psi \quad \text{and} \quad p_\psi = p_1 + p_2 , \tag{80}$$

so $p_\psi = p_1 + p_2 = k_1 + k_2$. Additionally, we pick up a factor of $-ig$ for each vertex, so the amplitude in the s-channel is

$$i\mathcal{A}_s = (-ig)^2 \frac{i}{(p_1 + p_2)^2 - m_\psi^2 + i\epsilon} . \tag{81}$$

Similarly, the t-channel process is shown in the right figure of Figure 3. Conserving momentum at each vertex yields:

$$k_1 = p_\psi + p_2 \quad \text{and} \quad k_2 + p_\psi = p_1 , \tag{82}$$

so $p_\psi = k_1 - p_2 = p_1 - k_2$.Additionally, we pick up a factor of $-ig$ for each vertex, so the amplitude in the s-channel is

$$i\mathcal{A}_t = (-ig)^2 \frac{i}{(k_1 - p_2)^2 - m_\psi^2 + i\epsilon} . \tag{83}$$

Summing these gives a consistent result with the Dyson method.

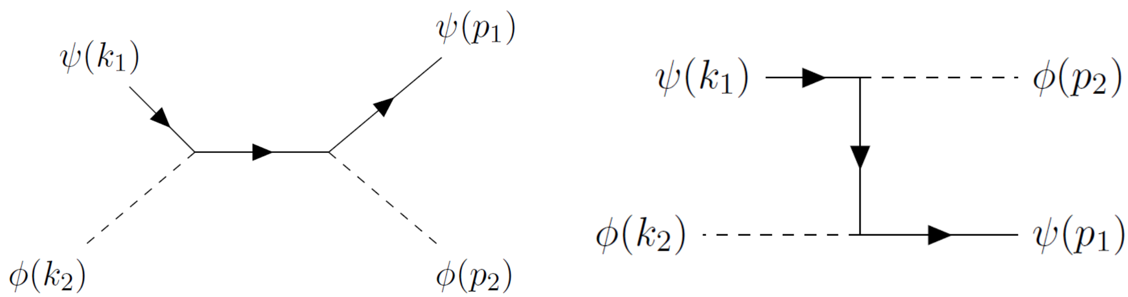

Figure 3: Feynman diagrams corresponding to the process $\psi(k_1) + \phi(k_2) \to \psi(p_1) + \phi(p_2)$.

F) Compute $i\mathcal{A}$ in the centre-of-mass frame for the scattering process $\psi(k_1) + \psi^\dagger(k_2) \to \phi(p_1) + \phi(p_2)$ using Dyson's formula. (Do not use any Feynman rules here!)

The matrix element for this process is given by

$$\langle \phi(p_1)\phi(p_2)| \frac{(-ig)^2}{2!} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathcal{T}\left\{\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\right\} |\psi(k_1)\psi^\dagger(k_2)\rangle \quad (84)$$

and again we see we must contract two fields to get non-vanishing terms. We cannot contract the real scalar fields because we need them to create the final state - so we are left with the four combinations shown in Equations 59-62. We are left with the same two redundant terms (killing the factor of 2 from the expansion of the exponential), and we get two ways of doing the contraction:

$$\langle p_1 p_2| : \psi_1 \phi_1 \psi_2^\dagger \phi_2 : |k_1 k_2^\dagger\rangle = e^{-ix_1\cdot k_1} e^{-ix_2\cdot k_2} e^{+ix_1\cdot p_1} e^{+ix_2\cdot p_2} \quad (85)$$

$$\langle p_1 p_2| : \psi_1 \phi_1 \psi_2^\dagger \phi_2 : |k_1 k_2^\dagger\rangle = e^{-ix_1\cdot k_1} e^{-ix_2\cdot k_2} e^{+ix_2\cdot p_1} e^{+ix_1\cdot p_2} \ . \quad (86)$$

We can use these to write the matrix element

$$(-ig)^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \psi_1^\dagger \psi_2 \langle p_1 p_2| : \psi_1 \phi_1 \psi_2^\dagger \phi_2 : |k_1 k_2^\dagger\rangle \ , \quad (87)$$

as

$$(-ig)^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \psi_1^\dagger \psi_2 \left\{ e^{-ix_1\cdot k_1} e^{-ix_2\cdot k_2} e^{+ix_1\cdot p_1} e^{+ix_2\cdot p_2} + e^{-ix_1\cdot k_1} e^{-ix_2\cdot k_2} e^{+ix_2\cdot p_1} e^{+ix_1\cdot p_2} \right\}$$

and inserting the complex scalar propagator:

$$(-ig)^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \frac{i e^{ip\cdot(x_1-x_2)}}{p^2 - m_\psi^2 + i\epsilon} \times \left\{ e^{-ix_1\cdot k_1} e^{-ix_2\cdot k_2} e^{+ix_1\cdot p_1} e^{+ix_2\cdot p_2} + e^{-ix_1\cdot k_1} e^{-ix_2\cdot k_2} e^{+ix_2\cdot p_1} e^{+ix_1\cdot p_2} \right\} \ . \quad (88)$$

Let's now combine the exponential factors:

$$e^{-ix_1\cdot(-p)} e^{-ix_2\cdot(p)} e^{-ix_1\cdot k_1} e^{-ix_2\cdot k_2} e^{-ix_1\cdot(-p_1)} e^{-ix_2\cdot(-p_2)} = e^{-ix_1\cdot(-p+k_1-p_1)} e^{-ix_2\cdot(p+k_2-p_2)} \quad (89)$$

$$e^{-ix_1\cdot(-p)} e^{-ix_2\cdot(p)} e^{-ix_1\cdot k_1} e^{-ix_2\cdot k_2} e^{-ix_2\cdot(-p_1)} e^{-ix_1\cdot(-p_2)} = e^{-ix_1\cdot(-p+k_1-p_2)} e^{-ix_2\cdot(p+k_2-p_1)} \ . \quad (90)$$

As in the previous process, we now integrate over both position-space coordinates:

$$\int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \to \delta^{(4)}([k_1 - p_1] - p)\delta^{(4)}(p - [p_2 - k_2]) \quad (91)$$

$$\int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \to \delta^{(4)}([k_1 - p_2] - p)\delta^{(4)}(p - [p_1 - k_2]) \ , \quad (92)$$

for the first and second terms, respectively. We are left with the integral over momentum-space:

$$i(-ig)^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4} \left\{ \frac{\delta^{(4)}([k_1 - p_1] - p)\delta^{(4)}(p - [p_2 - k_2])}{p^2 - m_\psi^2 + i\epsilon} + \frac{\delta^{(4)}([k_1 - p_2] - p)\delta^{(4)}(p - [p_1 - k_2])}{p^2 - m_\psi^2 + i\epsilon} \right\}$$

which we can carry out by killing one of the delta functions in each term. If we select the second delta function, we get the expression

$$i(-ig)^2 \left\{ \frac{\delta^{(4)}([k_1 - p_1] - [p_2 - k_2])}{[p_2 - k_2]^2 - m_\psi^2 + i\epsilon} + \frac{\delta^{(4)}([k_1 - p_2] - [p_1 - k_2])}{[p_1 - k_2]^2 - m_\psi^2 + i\epsilon} \right\} , \tag{93}$$

and if we select the first, we obtain

$$i(-ig)^2 \left\{ \frac{\delta^{(4)}([k_1 - p_1] - [p_2 - k_2])}{[k_1 - p_1]^2 - m_\psi^2 + i\epsilon} + \frac{\delta^{(4)}([k_1 - p_2] - [p_1 - k_2])}{[k_1 - p_2]^2 - m_\psi^2 + i\epsilon} \right\} , \tag{94}$$

and again we see all the delta functions are equivalent, which conserves momentum between the final and initial states. Using the definition of the amplitude, we are left with the results

$$\mathcal{A} = (-ig)^2 \left\{ \frac{1}{(p_2 - k_2)^2 - m_\psi^2 + i\epsilon} + \frac{1}{(p_1 - k_2)^2 - m_\psi^2 + i\epsilon} \right\} \tag{95}$$

$$= (-ig)^2 \left\{ \frac{1}{(k_1 - p_1)^2 - m_\psi^2 + i\epsilon} + \frac{1}{(k_1 - p_2)^2 - m_\psi^2 + i\epsilon} \right\} . \tag{96}$$

G) Draw the relevant Feynman diagram(s) for the process in (f). Then use Feynman rules to re-derive your answer in (f).

The s-channel is forbidden for this process, but the u-channel is not because the outgoing particles are identical. The t-channel diagram is shown in the left figure of Figure 4. Conserving momentum at each vertex yields:

$$k_1 = p_\psi + p_1 \quad \text{and} \quad k_2 + p_\psi = p_2 , \tag{97}$$

so $p_\psi = k_1 - p_1 = p_2 - k_2$. Additionally, we pick up a factor of $-ig$ for each vertex, so the amplitude in the s-channel is

$$i\mathcal{A}_t = (-ig)^2 \frac{i}{(k_1 - p_1)^2 - m_\psi^2 + i\epsilon} . \tag{98}$$

Similarly, the u-channel process is shown in the right figure of Figure 4. Conserving momentum at each vertex yields:

$$k_1 = p_\psi + p_2 \quad \text{and} \quad k_2 + p_\psi = p_1 , \tag{99}$$

so $p_\psi = k_1 - p_2 = p_1 - k_2$.Additionally, we pick up a factor of $-ig$ for each vertex, so the amplitude in the s-channel is

$$i\mathcal{A}_t = (-ig)^2 \frac{i}{(k_1 - p_2)^2 - m_\psi^2 + i\epsilon} . \tag{100}$$

Summing these gives a consistent result with the Dyson method.

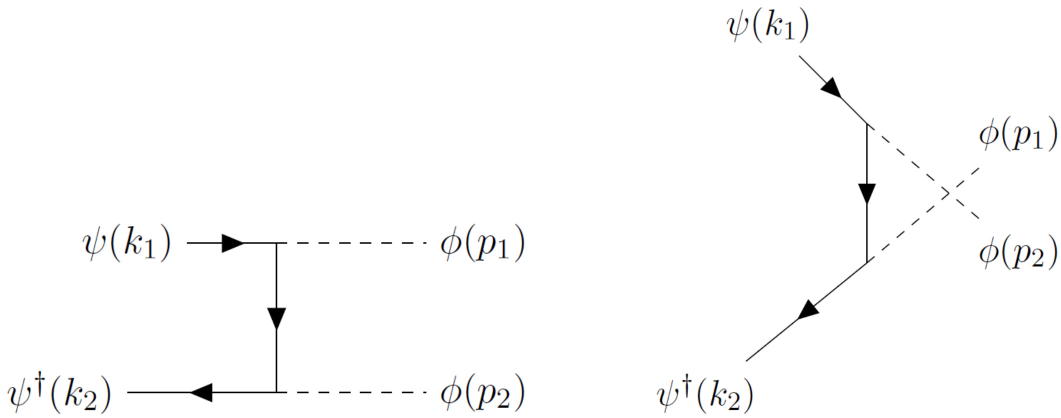

Figure 4: Feynman diagrams corresponding to the process $\psi(k_1) + \psi^\dagger(k_2) \to \phi(p_1) + \phi(p_2)$.

H) Assuming $m_\psi < m_\phi$, what is the minimal velocity of ψ in the centre-of-mass frame in order for process f to occur?

In the center of momentum[1]frame, the initial states' four-momenta are

$$k_1 = (E_\psi, \mathbf{v}) \quad \text{and} \quad k_2 = (E_{\psi^\dagger}, \mathbf{v}) \ , \tag{101}$$

where

$$E_\psi = \sqrt{m_\psi^2 + |\mathbf{v}|^2} \quad \text{and} \quad E_{\psi^\dagger} = \sqrt{m_{\psi^\dagger}^2 + (-1)^2|\mathbf{v}|^2} \ , \tag{102}$$

but $m_\psi = m_{\psi^\dagger}$, so $E_\psi = E_{\psi^\dagger}$. The total initial four-momentum is

$$k_1 + k_2 = (2\sqrt{m_\psi^2 + |\mathbf{v}|^2}, 0) \ . \tag{103}$$

If the incoming particles have just enough energy to make two real scalars, the real scalars are produced at rest, so

$$p_1 = p_2 = (m_\phi, 0) \ , \tag{104}$$

so the total final four-momentum is

$$p_1 + p_2 = (2m_\phi, 0) \ . \tag{105}$$

Conservation of momentum yields

$$2m_\phi = 2\sqrt{m_\psi^2 + |\mathbf{v}|^2} \ , \tag{106}$$

so solving for the magnitude of the center of mass velocity, we have

$$|\mathbf{v}| = \sqrt{m_\phi^2 - m_\psi^2} \ , \tag{107}$$

which seems like an obvious solution. If the incoming complex scalars have velocity higher than this, the real scalars will be produced with equal but opposite velocity (*i.e.*, back-to-back).

[1]Since the incoming and outgoing particles have the same mass, the center of mass frame is equivalent to the center of momentum frame.

I) Compute $i\mathcal{A}$ in the centre-of-mass frame for the scattering process $\phi(k) \to \psi(p_1) + \psi^\dagger(p_2)$ using Dyson's formula. (Do not use any Feynman rules here!)

This is a first-order process, so we must compute the matrix element:

$$(-ig)\int \mathrm{d}^4x \, \langle p_1 p_2^\dagger | \mathcal{T}\left\{\psi^\dagger\psi\phi\right\}|k\rangle \ , \tag{108}$$

where the time-ordered product is

$$\mathcal{T}\left\{\psi^\dagger\psi\phi\right\} =: \psi^\dagger\psi\phi : + \overbrace{\psi^\dagger\psi} : \phi : \ , \tag{109}$$

but the term with the only non-vanishing contraction does not have enough field operators to create/annihilate all the asymptotic states, and therefore this term vanishes as well. We must simply compute the matrix element:

$$\langle p_1 p_2^\dagger | : \psi^\dagger\psi\phi : |k\rangle \ , \tag{110}$$

which has only one valid (*i.e.*, non-vanishing) contraction:

$$\langle p_1 p_2^\dagger | : \psi^\dagger\psi\phi : |k\rangle = e^{+ix\cdot p_1} e^{+ix\cdot p_2} e^{-ix\cdot k} = e^{-ix\cdot(k-p_1-p_2)} \ . \tag{111}$$

Therefore, the matrix element for this process is

$$(-ig)\int \mathrm{d}^4x e^{-ix\cdot(k-p_1-p_2)} = (-ig)\delta^{(4)}(k - [p_1 + p_2]) \ , \tag{112}$$

and so the amplitude is

$$\mathcal{A} = -g \ , \tag{113}$$

from the definition (Equation 3).

J) Draw the relevant Feynman diagram(s) for the process in (i). Then use Feynman rules to re-derive your answer in (i).

The only (to leading order) diagram for this process is shown in Figure 5. There is one vertex, so we pick up a factor of $-ig$, and by Feynman's rules, we have

$$i\mathcal{A} = -ig \ , \tag{114}$$

which is consistent with Dyson's method.

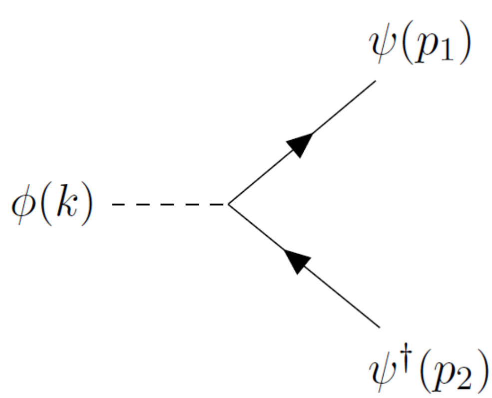

Figure 5: Feynman diagram corresponding to the process $\phi(k) \to \psi(p_1) + \psi^\dagger(p_2)$.

At next-to-leading order, this is a third order process, and can be represented by the diagram shown in Figure 6. This is a loop diagram and will result in a divergent momentum integral.

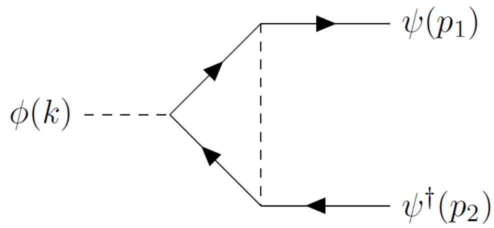

Figure 6: The next-to-leading order Feynman diagram corresponding to process $\phi(k) \to \psi(p_1) + \psi^\dagger(p_2)$.

K) Can the process in (i) occur for arbitrary masses m_ϕ and m_ψ?

No, in the center of mass frame, the real scalar has no velocity and thus its four-momentum is $k = (m_\phi, 0)$. In order to create two complex scalars at rest, i must be that $m_\phi = 2m_\psi$. To create two complex scalars with relative velocities, it must be that $m_\phi > 2m_\psi$, so

$$m_\phi \geq 2m_\psi \ . \tag{115}$$

L) Can the process $\psi(k_1) + \psi(k_2) \to \psi^\dagger(p_1) + \psi^\dagger(p_2)$ occur? Why or why not?

No, this is an impossible process because $U(1)$ charge cannot be conserved! The initial states carry a $U(1)$ charge of $+2$ and the final states carry a $U(1)$ charge of -2.