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Quantum Field Theory II
QFT and the Standard Model - M. Schwartz
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1 Classical Electromagnetism.

Consider classical electromagnetism, defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{1}$$

(a) Derive Maxwell's equations by computing the Euler-Lagrange equations of motion, and identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B_k = -F^{ij}$.

First we note the form of the field tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2}$$

Using the fact $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we see that the Lagrangian only depends on partial derivatives of the photon field:

$$\mathcal{L} = -\frac{1}{4} \left(\partial_\mu A_\nu F^{\mu\nu} - \partial_\nu A_\mu F^{\mu\nu} \right) \ , \tag{3}$$

on the second term we can switch the dummy indeces and use the fact that $F^{\mu\nu}$ is antisymmetric:

$$\mathcal{L} = -\frac{1}{2} \left(\partial_\mu A_\nu F^{\mu\nu} \right) \ . \tag{4}$$

Since there is no dependence on the field A^μ (outside of partial derivatives) the Euler-Lagrange equation gives us

$$-\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} = 0 \ . \tag{5}$$

So now we take the derivative indicated above:

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} = -\frac{1}{2} F^{\mu\nu} \ . \tag{6}$$

When we took the derivative we neglected the fact that we could have instead written the second field tensor out performed the derivative. This would result in an identical term, so we are free to cancel the factor of 2. The negative sign is killed by the equation of motion, and we obtain

$$0 = \partial_\mu F^{\mu\nu} \ . \tag{7}$$

We can do a similar derivation with the dual of the field strength tensor $\mathscr{F}^{\mu\nu}$, which by definition is

$$\mathscr{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \ , \tag{8}$$

and one can see it satisifes the same equation of motion:

$$0 = \partial_\mu \mathscr{F}^{\mu\nu} \ . \tag{9}$$

The system of these equations of motion results in the sourceless Maxwell equations (*i.e.*, all the currents are zero). Note that both $F^{\mu\nu}$ and its dual are antisymmetric, so

$$0 = \partial_i F^{i0} = -\partial_i F^{0i} = -\partial_1(-E_x) - \partial_2(-E_y) - \partial_3(-E_v) = \boldsymbol{\nabla}\cdot\mathbf{E} \ . \tag{10}$$

We can take a similar approach:

$$0 = \partial_\nu F^{\nu i} = \partial_0 F^{0i} - \partial_j(F^{ji}) = \partial_0(-E^i) + \partial_j(\epsilon^{ijk}B_k) = \frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times\mathbf{B} \ . \tag{11}$$

Doing the same with the dual gives

$$0 = \partial_i \mathscr{F}^{i0} = -\partial_i \mathscr{F}^{0i} = -\partial_i(-B_i) = \boldsymbol{\nabla}\cdot\mathbf{B} \tag{12}$$

$$0 = \partial_\mu \mathscr{F}^{\mu i} = \partial_0 \mathscr{F}^{0i} - \partial_j \mathscr{F}^{ji} = -\partial_0 B_i \partial_j(-F^{ij}) = -\partial_0 B_i + \partial_j(\epsilon^{ijk}E_k) = \frac{\partial \mathbf{B}}{\partial t} - \boldsymbol{\nabla}\times\mathbf{E} \ . \tag{13}$$

(b) Construct the energy-momentum tensor for this theory. You will need to supplement the naively computed $T^{\mu\nu}$ in the following way:

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} \ , \tag{14}$$

where $K^{\rho\mu\nu} = F^{\mu\rho}A^\nu$. This is needed because $T^{\mu\nu}$ is not symmetric under interchange of its indices, which should occur for the energy-momentum tensor. Show that $K^{\rho\mu\nu}$, so that it does not affect the conservation of energy-momentum. Show that $\hat{T}^{\mu\nu}$ is symmetric, and that it leads to the usual expressions for the energy density and momentum density in electromagnetic fields:

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2), \quad \mathbf{S} = \mathbf{E}\times\mathbf{B} \ . \tag{15}$$

For the energy-momentum tensor to be symmetric, we require

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\ \nu} = T^{\mu\nu} + \partial_\rho(F^{\mu\rho}A^\nu) = T^{\mu\nu} - \partial_\rho(F^{\rho\mu}A^\nu) \tag{16}$$

$$\hat{T}^{\nu\mu} = T^{\nu\mu} + \partial_\rho K^{\rho\nu\ \mu} = T^{\nu\mu} + \partial_\rho(F^{\nu\rho}A^\mu) = T^{\nu\mu} - \partial_\rho(F^{\rho\nu}A^\mu) \tag{17}$$

to be equivalent. Let's investigate the terms of the form

$$\partial_\rho(F^{\rho\mu}A^\nu) = A^\nu \partial_\rho F^{\rho\mu} + F^{\rho\mu}\partial_\rho A^\nu = F^{\rho\mu}\partial_\rho A^\nu \ , \tag{18}$$

from using the equations of motion $\partial_\alpha F^{\alpha\beta} = 0$. We have then,

$$\hat{T}^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu = T^{\mu\nu} + F^{\mu\rho}\partial_\rho A^\nu \tag{19}$$

$$\hat{T}^{\nu\mu} = T^{\nu\mu} - F^{\rho\nu}\partial_\rho A^\mu = T^{\nu\mu} + F^{\nu\rho}\partial_\rho A^\mu \ . \tag{20}$$

From Schwartz 3.35, the energy-momentum tensor is

$$T_{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)}\partial_\nu \phi_n - g_{\mu\nu}\mathcal{L} \ , \tag{21}$$

so for us this is

$$T_{\mu\nu} = -F_{\mu\nu}\partial_\nu A^\mu - g_{\mu\nu}\mathcal{L} \ . \tag{22}$$

2 Scalar QED.

Consider the scalar QED theory defined in class. Denoting the positively and negatively charged scalar particles respectively as $\tilde{e}^-$ and $\tilde{e}^+$.

(a) Compute the amplitudes for the scattering processes $\tilde{e}^-\tilde{e}^- \to \tilde{e}^-\tilde{e}^-$ and $\tilde{e}^-\tilde{e}^+ \to \tilde{e}^-\tilde{e}^+$.

The Feynman rules we will need for these processes are:

- Photon propagator:
$$\frac{-i}{p^2 + i\epsilon}\left\{g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right\}$$
- Vertex (two charged scalars A,C and one photon b):
$$-ie(p_A + p_C)_\mu$$

2.1 $\tilde{e}^-(p_1)\ \tilde{e}^-(p_2) \to \tilde{e}^-(p_3)\ \tilde{e}^-(p_4)$

To leading order, this process can occur through the t and u channels, and the corresponding diagrams shown in figure 1.

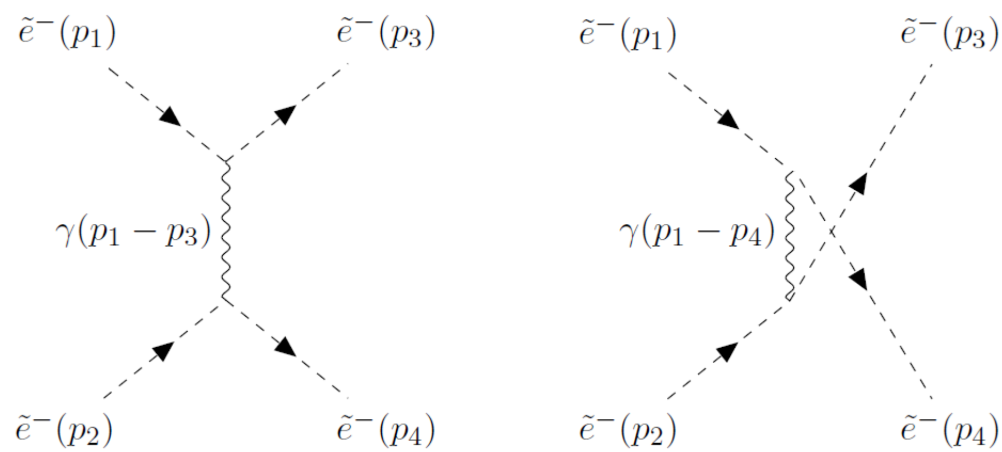

Figure 1: The Feynman diagrams describing the process $\tilde{e}^-\tilde{e}^- \to \tilde{e}^-\tilde{e}^-$. (Left) t-channel. (Right) u-channel.

2.1.1 t-Channel Matrix Element

We can write down the matrix element for the t-channel:

$$i\mathcal{M}_t = (-ie)^2(p_1+p_3)_\mu \frac{-i}{p^2+i\epsilon}\left\{g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right\}(p_2+p_4)_\nu\ , \tag{23}$$

where the photon momentum $p = p_1 - p_3 = p_4 - p_2$. We can contract the index ν:

$$i\mathcal{M}_t = (p_1+p_3)_\mu \frac{ie^2}{(p_1-p_3)^2}\left\{(p_2+p_4)^\mu - (1-\xi)\frac{(p_1-p_3)^\mu (p_1-p_3)\cdot(p_2+p_4)}{p^2}\right\}\ , \tag{24}$$

note we've taken the $\epsilon \to 0$ limit. Lets consider the first term:

$$i\mathcal{M}_{t-1} = \frac{ie^2}{(p_3-p_1)^2}(p_1+p_3)_\mu(p_2+p_4)^\mu \ , \tag{25}$$

where, in the center-of-momentum frame:

$$p_1^\mu = (E_i, \mathbf{p}_i) \qquad\qquad p_3^\mu = (E_f, \mathbf{p}_f) \tag{26}$$
$$p_2^\mu = (E_i, -\mathbf{p}_i) \qquad\qquad p_4^\mu = (E_f, -\mathbf{p}_f) \ , \tag{27}$$

so

$$Q \equiv (p_1+p_3)_\mu(p_2+p_4)^\mu = (E_i+E_f)^2 - (\mathbf{p}_i+\mathbf{p}_f)\cdot(-\mathbf{p}_i-\mathbf{p}_f) \tag{28}$$
$$= (E_i+E_f)^2 + |\mathbf{p}_i|^2 + |\mathbf{p}_f|^2 + 2|\mathbf{p}_i|\mathbf{p}_f|\cos\theta \ , \tag{29}$$

where θ is the scattering angle between p_1 and p_3 (or equivalently p_2 and p_4). We have also

$$t \equiv (p_1-p_3)^2 = (p_3-p_1)^2 = (E_f-E_i)^2 - (\mathbf{p}_f-\mathbf{p}_i)\cdot(\mathbf{p}_f-\mathbf{p}_i) \tag{30}$$
$$= E_f^2 + E_i^2 + 2E_fE_i - \left(|\mathbf{p}_f|^2 + |\mathbf{p}_i|^2 - 2|\mathbf{p}_f||\mathbf{p}_i|\cos\theta\right) \tag{31}$$
$$= 2m_{\tilde{e}}^2 + 2E_fE_i + 2|\mathbf{p}_f||\mathbf{p}_i|\cos\theta \ . \tag{32}$$

Let us expand on equation 29:

$$Q = (E_i+E_f)^2 + (E_i^2 - m_{\tilde{e}}^2) + (E_f^2 - m_{\tilde{e}}^2) + 2\sqrt{(E_f^2-m_{\tilde{e}}^2)(E_i^2-m_{\tilde{e}}^2)}\cos\theta \tag{33}$$
$$= 2E_i^2 + 2E_f^2 - 2m_{\tilde{e}}^2 + 2E_iE_f + 2\sqrt{(E_f^2-m_{\tilde{e}}^2)(E_i^2-m_{\tilde{e}}^2)}\cos\theta \ , \tag{34}$$

we can now identify $s = 4E_i^2 = 4E_f^2$ (the square of the total energy available), such that

$$Q = 2\left\{\frac{s}{4} + \frac{s}{4} - m_{\tilde{e}}^2 + E_iE_f + \sqrt{(E_f^2-m_{\tilde{e}}^2)(E_i^2-m_{\tilde{e}}^2)}\cos\theta\right\} \ . \tag{35}$$

Note we are dealing with identical particles in the center-of-mass frame, and as such $E_i = E_f$:

$$Q = 2\left\{\frac{s}{4} + \frac{s}{4} - m_{\tilde{e}}^2 + \frac{s}{4} + (E_i^2 - m_{\tilde{e}}^2)\cos\theta\right\} \tag{36}$$
$$= 2\left\{\frac{3s}{4} - m_{\tilde{e}}^2 + \left(\frac{s}{4} - m_{\tilde{e}}^2\right)\cos\theta\right\} \tag{37}$$
$$= \frac{1}{2}s(3+\cos\theta) - 2m_{\tilde{e}}^2(1+\cos\theta) \ . \tag{38}$$

Therefore, the first term in the matrix element is

$$i\mathcal{M}_{t-1} = \frac{ie^2}{t}Q = (ie^2)\frac{s(3+\cos\theta) - 4m_{\tilde{e}}^2(1+\cos\theta)}{2t} \ . \tag{39}$$

We can argue the second term:

$$i\mathcal{M}_{t-2} = \frac{-ie^2(1-\xi)}{(p_1-p_3)^2 + i\epsilon}(p_1+p_3)_\mu\frac{(p_1-p_3)^\mu(p_1-p_3)^\nu}{p^2}(p_2+p_4)_\nu \ , \tag{40}$$

vanishes because

$$(p_1+p_3)_\mu(p_1-p_3)^\mu = (E_i+E_f, \mathbf{p}_i+\mathbf{p}_f)\cdot(E_i-E_f, \mathbf{p}_i-\mathbf{p}_f) \tag{41}$$
$$= (2E_i)(0) - (\mathbf{p}_i+\mathbf{p}_f)\cdot(\mathbf{p}_i-\mathbf{p}_f) \tag{42}$$
$$= (\mathbf{p}_i+\mathbf{p}_f)\cdot(\mathbf{p}_f-\mathbf{p}_i) = |\mathbf{p}_f|^2 - |\mathbf{p}_i|^2 \tag{43}$$
$$= (E_f^2 - m_{\tilde{e}}^2) - (E_i^2 - m_{\tilde{e}}^2) = 0 \ , \tag{44}$$

and thus the whole term vanishes.

2.1.2 u-Channel Matrix Element

The matrix element for this channel is

$$i\mathcal{M}_u = (-ie)^2 (p_1 + p_4)_\mu \frac{-i}{p^2 + i\epsilon} \left\{ g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2} \right\} (p_2 + p_3)_\nu \ , \tag{45}$$

with $p = p_1 - p_4$. First, let us argue the second term in the photon propagator vanishes here as well. This term is proportional to the Lorentz invariant quantity

$$(p_1 + p_4)_\mu (p_4 - p_1)^\mu = (E_1 + E_4, \mathbf{p}_1 + \mathbf{p}_4) \cdot (E_4 - E_1, \mathbf{p}_4 - \mathbf{p}_1) \ , \tag{46}$$

we are free to evaluate this in any frame, so we select the center of mass frame: $E_1 \equiv E_i = E_4 \equiv E_f$, so

$$\begin{aligned} (p_1 + p_4)_\mu (p_4 - p_1)^\mu &= (2E_i, \mathbf{p}_i + \mathbf{p}_f) \cdot (0, \mathbf{p}_f - \mathbf{p}_i) = -\left\{ |\mathbf{p}_f|^2 - |\mathbf{p}_i|^2 \right\} & (47) \\ &= (E_f^2 - m_{\tilde{e}}^2) - (E_i^2 - m_{\tilde{e}}^2) = 0 \ , & (48) \end{aligned}$$

and we can see that as long as the incoming and outgoing scalar has the same mass, this term will always vanish - we will use this result in the next section. The matrix element is then

$$i\mathcal{M}_u = (-ie)^2 (p_1 + p_4)_\mu \frac{-i}{(p_1 - p_4)^2 + i\epsilon} (p_2 + p_3)^\mu \ , \tag{49}$$

which is proportional to the dot-product:

$$(p_1 + p_4)_\mu (p_2 + p_3)^\mu = (2E, \mathbf{p}_i - \mathbf{p}_f) \cdot (2E, -\mathbf{p}_i + \mathbf{p}_f) \ , \tag{50}$$

in the center-of-momentum frame, using the same definitions as before, with $E = E_i = E_f$. Carrying out the product yields

$$\begin{aligned} (p_1 + p_4)_\mu (p_2 + p_3)^\mu &= 4E^2 - \left\{ -|\mathbf{p}_f|^2 - |\mathbf{p}_f|^2 + 2|\mathbf{p}_i||\mathbf{p}_f| \cos\theta \right\} & (51) \\ &= s + (E^2 - m_{\tilde{e}}^2) + (E^2 - m_{\tilde{e}}^2) - 2\sqrt{(E^2 - m_{\tilde{e}}^2)(E^2 - m_{\tilde{e}}^2)} \cos\theta & (52) \\ &= s + \frac{s}{2} - 2m_{\tilde{e}}^2 - 2E^2 \cos\theta - 2m_{\tilde{e}}^2 \cos\theta & (53) \\ &= \frac{s}{2}(3 - \cos\theta) - 2m_{\tilde{e}}^2 (1 - \cos\theta) \ . & (54) \end{aligned}$$

Inserting this into the matrix element, and performing some simplification yields

$$i\mathcal{M}_u = \frac{ie^2}{u} \frac{s}{2}(3 - \cos\theta) - 2m_{\tilde{e}}^2 (1 - \cos\theta) \ . \tag{55}$$

2.1.3 Squaring the Amplitude

We will collect our results here:

$$\begin{aligned} i\mathcal{M}_t &= ie^2 \frac{(p_1 + p_3) \cdot (p_2 + p_4)}{t} = (ie^2) \frac{s(3 + \cos\theta) - 4m_{\tilde{e}}^2 (1 + \cos\theta)}{2t} & (56) \\ i\mathcal{M}_u &= ie^2 \frac{(p_1 + p_4) \cdot (p_2 + p_3)}{u} = (ie^2) \frac{s(3 - \cos\theta) - 4m_{\tilde{e}}^2 (1 - \cos\theta)}{2u} \ . & (57) \end{aligned}$$

We can examine the dot products:

$$(p_1 + p_3) \cdot (p_2 + p_4) = p_1 \cdot p_2 + p_1 \cdot p_4 + p_3 \cdot p_2 + p_3 \cdot p_4 \tag{58}$$

We must note that since the final state particles are identical, the matrix elements have a relative sign difference, so the total matrix element for this process is

$$i\mathcal{M} = i\mathcal{M}_t - i\mathcal{M}_u \ , \tag{59}$$

so the modulus squared is

$$|\mathcal{M}|^2 = |\mathcal{M}_t|^2 + |\mathcal{M}_u|^2 + 2\Re\{\mathcal{M}_t^*\mathcal{M}_u\} \ . \tag{60}$$

There is no spin or external polarization in this process, so we do not need to sum over final states or average over initial states. Let us calculate these terms:

$$|\mathcal{M}_t|^2 = e^4 \left(\frac{s(3+\cos\theta) - 4m_{\tilde{e}}^2(1+\cos\theta)}{2t} \right)^2 = \frac{e^4}{4t^2} \left(s(3+\cos\theta) - 4m_{\tilde{e}}^2(1+\cos\theta) \right)^2 \tag{61}$$

$$|\mathcal{M}_u|^2 = e^4 \left(\frac{s(3-\cos\theta) - 4m_{\tilde{e}}^2(1-\cos\theta)}{2u} \right)^2 = \frac{e^4}{4u^2} \left(s(3-\cos\theta) - 4m_{\tilde{e}}^2(1-\cos\theta) \right)^2 \tag{62}$$

$$\mathcal{M}_t^*\mathcal{M}_u = e^4 \frac{s(3+\cos\theta) - 4m_{\tilde{e}}^2(1+\cos\theta)}{2t} \frac{s(3-\cos\theta) - 4m_{\tilde{e}}^2(1-\cos\theta)}{2u} \tag{63}$$

$$= \frac{e^4}{4tu} \left(s(3+\cos\theta) - 4m_{\tilde{e}}^2(1+\cos\theta) \right) \left(s(3-\cos\theta) - 4m_{\tilde{e}}^2(1-\cos\theta) \right) \ . \tag{64}$$

2.2 $\tilde{e}^-(p_1)\ \tilde{e}^+(p_2) \to \tilde{e}^-(p_3)\ \tilde{e}^+(p_4)$

This process can occur through the t and u channels, and the corresponding diagrams shown in figure 2. As stated previously, the second term in the photon propagator vanishes in the center of momentum frame, thus due to Lorentz invariance, vanishes in all inertial frame for particles with identical mass. We have that the modulus squared of the matrix element for the entire process is

$$|\mathcal{M}|^2 = |\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 + 2\Re\{\mathcal{M}_s^*\mathcal{M}_t\} \ , \tag{65}$$

so we will calculate these terms separately below. Now since we are dealing with anti-particles as well, if the momentum is in the opposite direction of the charge flow, we pick up a negative sign.

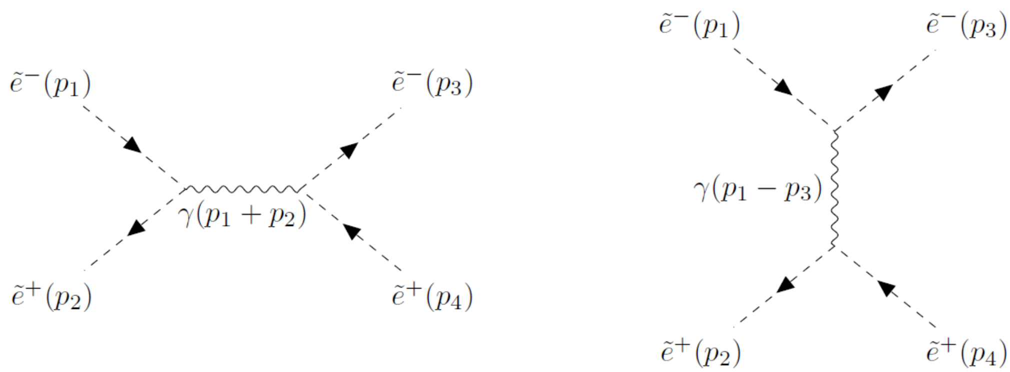

Figure 2: The Feynman diagrams describing the process $\tilde{e}^-\tilde{e}^+ \to \tilde{e}^-\tilde{e}^+$. (Left) s-channel. (Right) t-channel.

2.2.1 s-Channel Matrix Element

The matrix element for this channel is

$$i\mathcal{M}_s = (p_1 - p_2)_\mu \frac{ie^2 g^{\mu\nu}}{p^2 + i\epsilon}(p_3 - p_4)_\nu = \frac{ie^2}{(p_1+p_2)^2}(p_1-p_2)_\mu(p_3-p_4)^\mu \ , \tag{66}$$

where the propagator momentum is $p = p_1 + p_2$. We will continue working in the center of mass frame, using the definitions given by equations 26 and 27, with $E_f = E_i \equiv E$ and $|\mathbf{p}_f| = |\mathbf{p}_i| \equiv p$. The product of the momenta is

$$(p_1 - p_2)_\mu(p_3 - p_4)^\mu = (0, 2\mathbf{p}_i)\cdot(0, 2\mathbf{p}_f) = -4\mathbf{p}_i\cdot\mathbf{p}_f = -4p^2\cos\theta \tag{67}$$

$$= -4(E^2 - m_{\tilde{e}}^2)\cos\theta = (4m_{\tilde{e}}^2 - s)\cos\theta \tag{68}$$

We can then express the matrix element as

$$i\mathcal{M}_s = \frac{ie^2}{s}(p_1-p_2)_\mu(p_3-p_4)^\mu = \frac{ie^2}{s}(4m_{\tilde{e}}^2 - s)\cos\theta \ . \tag{69}$$

2.2.2 t-Channel Matrix Element

The matrix element for this channel is

$$i\mathcal{M}_t = \frac{ie^2}{(p_1 - p_3)^2 + i\epsilon}(p_1 + p_3)_\mu(-p_2 - p_4)_\mu = -\frac{ie^2}{t}(p_1 + p_3)_\mu(p_2 + p_4)^\mu \ , \tag{70}$$

the scalar product is

$$(p_1 + p_3)_\mu(p_2 + p_4)^\mu = (2E, \mathbf{p}_i + \mathbf{p}_f) \cdot (2E, -\mathbf{p}_i - \mathbf{p}_f) = 4E^2 + (\mathbf{p}_i + \mathbf{p}_f)^2 \tag{71}$$

$$= 4E^2 + p^2 + p^2 + 2p^2 \cos\theta = s + 2p^2(1 + \cos\theta) \tag{72}$$

$$= s + 2(E^2 - m_{\tilde{e}}^2)(1 + \cos\theta) = \frac{s}{2}(2 - \cos\theta) - 2m_{\tilde{e}}^2 \cos\theta \tag{73}$$

$$= \frac{s}{2}(3 + \cos\theta) - 2m_{\tilde{e}}^2(1 + \cos\theta) \ . \tag{74}$$

The matrix element becomes

$$i\mathcal{M}_t = -\frac{ie^2}{2t}\left\{s(3 + \cos\theta) - 4m_{\tilde{e}}^2(1 + \cos\theta)\right\} \ . \tag{75}$$

2.2.3 Squaring the Amplitude

It is trivial to take the modulus squared of the terms:

$$|\mathcal{M}_s|^2 = \frac{e^4}{s}(4m_{\tilde{e}}^2 - s)^2 \cos^2\theta \tag{76}$$

$$|\mathcal{M}_t|^2 = \frac{e^4}{4t^2}\left\{s(3 + \cos\theta) - 4m_{\tilde{e}}^2(1 + \cos\theta)\right\}^2 \tag{77}$$

$$2\Re\{\mathcal{M}_s^*\mathcal{M}_t\} = -\frac{e^4}{st}(4m_{\tilde{e}}^2 - s)(\cos\theta)\left\{s(3 + \cos\theta) - 4m_{\tilde{e}}^2(1 + \cos\theta)\right\} \ . \tag{78}$$

(b) Take the non-relativistic limits of the amplitudes computed in part (a), and compare them to the usual quantum-mechanical scattering expressions. What are the non-relativistic potentials that mediate the scatterings in part (a)?

Let us collect the results from section 5:

$$\tilde{e}^-(p_1)\ \tilde{e}^-(p_2) \to \tilde{e}^-(p_3)\ \tilde{e}^-(p_4)$$

$$i\mathcal{M}_t = -ie^2 \frac{2+\beta^2(1+\cos\theta)}{\beta^2\,(1-\cos\theta)} \tag{79}$$

$$i\mathcal{M}_u = -ie^2 \frac{2+\beta^2\sin^2\theta}{\beta^2(1+\cos\theta)} \tag{80}$$

$$\tilde{e}^-(p_1)\ \tilde{e}^+(p_2) \to \tilde{e}^-(p_3)\ \tilde{e}^+(p_4)$$

$$i\mathcal{M}_s = ie^2 \frac{2+\beta^2\sin^2\theta}{2} \tag{81}$$

$$i\mathcal{M}_t = ie^2 \frac{2+\beta^2(1+\cos\theta)}{\beta^2\,(1-\cos\theta)} \tag{82}$$

where

$$\beta = \pm\sqrt{1-\frac{4m_{\tilde{e}}^2}{E_{cm}^2}}\ , \tag{83}$$

with $|\beta| < 1$, and $E_{cm} = 2E_i$. In the non-relativistic limit $\beta \ll 1$.

Let us sum the terms for the first process:

$$\begin{aligned}
\mathcal{M} &= e^2\left\{\frac{2+\beta^2 1+\cos\theta}{\beta^2 1-\cos\theta} + \frac{2+\beta^2\sin^2\theta}{\beta^2 1+\cos\theta}\right\} && (84)\\
&= e^2\frac{\beta^2\sin^2\theta+\beta^2\cos^2\theta+2\beta^2\cos\theta-\beta^2\sin^2\theta\cos\theta+\beta^2+4}{\beta^2(1+\cos\theta)(1-\cos\theta)} && (85)\\
&= e^2\frac{\beta^2(1+2\cos\theta-\sin^2\theta\cos\theta+1)+4}{\beta^2(1+\cos\theta)(1-\cos\theta)} && (86)\\
&= \frac{e^2}{1-\cos^2\theta}\left(\frac{4}{\beta^2}+2+2\cos\theta-\sin^2\theta\cos\theta\right) && (87)\\
&= \frac{e^2}{\sin^2\theta}\left(\frac{4}{\beta^2}+2+2\cos\theta-(1-\cos^2\theta)\cos\theta\right) && (88)\\
&= \frac{e^2}{\sin^2\theta}\left(\frac{4}{\beta^2}+2+\cos\theta+\cos^3\theta\right)\ , && (89)
\end{aligned}$$

and the second:

$$\mathcal{M} = e^2 \left\{ \frac{2 + \beta^2 \sin^2\theta}{2} + \frac{2 + \beta^2 (1 + \cos\theta)}{\beta^2 \, (1 - \cos\theta)} \right\} \tag{90}$$

$$= e^2 \left\{ \frac{(2 + \beta^2 \sin^2\theta)(\beta^2 \, (1 - \cos\theta))}{2\beta^2 \, (1 - \cos\theta)} + \frac{4 + 2\beta^2 (1 + \cos\theta)}{2\beta^2 \, (1 - \cos\theta)} \right\} \tag{91}$$

$$= \frac{e^2}{1 - \cos\theta} \left(\frac{4}{\beta^2} + 4 + \beta^2 (1 - \cos\theta) \sin^2\theta \right) \ . \tag{92}$$

In the $\beta \ll 1$ limit, these become

$$\mathcal{M} = \frac{4e^2}{\beta^2 \sin^2\theta} \tag{93}$$

$$\mathcal{M} = \frac{4e^2}{\beta^2 (1 - \cos\theta)} \ , \tag{94}$$

respectively.

3 Electron Spin & Lorentz Boost.

Consider a electron at rest, with spin along the $+z$ axis. Its wave function can be written as

$$\psi(x) = u(p)e^{ip\cdot x} = \sqrt{m}\begin{pmatrix}\xi_+ \\ \xi_+\end{pmatrix} \ . \tag{95}$$

The spin and boost operators for this representation can be written as

$$S^i = \frac{1}{4}\epsilon^{ijk}\epsilon^{jkl}\begin{pmatrix}\sigma^l & 0 \\ 0 & \sigma^l\end{pmatrix}, \quad K^i = -\frac{i}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix} \ , \tag{96}$$

respectively. A Lorentz transformation acts on this particle as

$$\psi \to \Lambda_{1/2}\psi$$

where

$$\Lambda_{1/2} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \ . \tag{97}$$

(a) What is the expectation value of S^z in the at-rest electron state (be careful with the normalization of the single-particle state)?

Consider the S^z operator:

$$S^3 = \frac{1}{4}\epsilon^{3jk}\epsilon^{jkl}\begin{pmatrix}\sigma^l & 0 \\ 0 & \sigma^l\end{pmatrix} = \frac{1}{4}\left\{\epsilon^{312}\epsilon^{12l}\begin{pmatrix}\sigma^l & 0 \\ 0 & \sigma^l\end{pmatrix} + \epsilon^{321}\epsilon^{21l}\begin{pmatrix}\sigma^l & 0 \\ 0 & \sigma^l\end{pmatrix}\right\} \ , \tag{98}$$

since the completely antisymmetric tensor is zero for any repeated indeces. We can permute one of these tensors, noting $\epsilon^{312} = 1 = -\epsilon^{321}$:

$$S^3 = \frac{1}{4}\left\{\epsilon^{12l}\begin{pmatrix}\sigma^l & 0 \\ 0 & \sigma^l\end{pmatrix} - \epsilon^{21l}\begin{pmatrix}\sigma^l & 0 \\ 0 & \sigma^l\end{pmatrix}\right\} = \frac{1}{4}(2\epsilon^{12l})\left\{\begin{pmatrix}\sigma^l & 0 \\ 0 & \sigma^l\end{pmatrix}\right\} \tag{99}$$

$$= \frac{1}{4}(2\epsilon^{123})\left\{\begin{pmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix} \ . \tag{100}$$

To find the expectation value of this operator, we calculate the matrix element

$$\langle\psi|S^3|\psi\rangle = \frac{1}{2E}\frac{1}{2}\psi^\dagger\begin{pmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}\psi = \frac{1}{2E}\frac{m}{2}\begin{pmatrix}\xi^\dagger & \xi^\dagger\end{pmatrix}\begin{pmatrix}\sigma^3\xi \\ \sigma^3\xi\end{pmatrix} = \frac{1}{2E}m\xi^\dagger\sigma^3\xi \ , \tag{101}$$

where the factor of $2E$ comes in from the relativistic normalization $\psi^\dagger\psi = 2E$. In the rest frame of the electron $E = m_e$, so

$$\langle\psi|S^3|\psi\rangle = \frac{1}{2}\xi^\dagger\sigma^3\xi \ , \tag{102}$$

We can define a two-component spinor ξ as

$$\xi = N\begin{pmatrix}a \\ b\end{pmatrix} \ , \tag{103}$$

where N is a normalization factor, such that

$$\langle\psi|S^3|\psi\rangle = \frac{1}{2}|N|^2 \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}|N|^2(a^2 - b^2) \tag{104}$$

We can determine the value for $|N|^2$ from the normalization conditions

$$1 = \xi^\dagger \xi = |N|^2(a^2 + b^2) \ , \tag{105}$$

so

$$\langle\psi|S^3|\psi\rangle = \frac{1}{2}\frac{a^2 - b^2}{a^2 + b^2} \ . \tag{106}$$

Outside of the normalization condition ξ is arbitrary, but we are interested with spin oriented along the $+z$ axis so we will take $a = 1$ and $b = 0$:

$$\langle\psi|S^3|\psi\rangle = \frac{1}{2} \ . \tag{107}$$

(b) Perform a boost by an amount η along the x-direction. What is the new wavefunction of the electron? What is the expectation value of S^z in this state, and what does it become in the ultra-relativistic ($\eta \to \infty$) limit?

A boost along the x axis by η of the wavefunction yields

$$\psi(x) \to \exp\left\{ -\frac{1}{2}\eta \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \right\} \sqrt{m} \begin{pmatrix} \xi_+ \\ \xi_+ \end{pmatrix} \ , \tag{108}$$

see Peskin 3.49. Following Peskin, this can be expressed

$$\psi'(x) = \left\{ \cosh(\eta/2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sinh(\eta/2) \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \right\} \sqrt{m} \begin{pmatrix} \xi_+ \\ \xi_+ \end{pmatrix} \tag{109}$$

$$= \sqrt{m} \left\{ \cosh(\eta/2) \begin{pmatrix} \xi_+ \\ \xi_+ \end{pmatrix} - \sinh(\eta/2) \begin{pmatrix} \sigma^1\xi_+ \\ \sigma^1\xi_+ \end{pmatrix} \right\} \ , \tag{110}$$

let us note

$$\sigma^1 \xi_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \xi_- \ , \tag{111}$$

so

$$\psi'(x) = \sqrt{m} \left\{ \cosh(\eta/2) \begin{pmatrix} \xi_+ \\ \xi_+ \end{pmatrix} - \sinh(\eta/2) \begin{pmatrix} \xi_- \\ \xi_- \end{pmatrix} \right\} \ . \tag{112}$$

Let us act the S_z operator on the new wavefunction:

$$S^3 \left|\psi'_+\right\rangle = \frac{\sqrt{m}}{2} \left\{ \cosh(\eta/2) \begin{pmatrix} \sigma^3\xi_+ \\ \sigma^3\xi_+ \end{pmatrix} - \sinh(\eta/2) \begin{pmatrix} \sigma^3\xi_- \\ \sigma^3\xi_- \end{pmatrix} \right\} \tag{113}$$

$$= \frac{\sqrt{m}}{2} \left\{ \cosh(\eta/2) \begin{pmatrix} \xi_+ \\ \xi_+ \end{pmatrix} + \sinh(\eta/2) \begin{pmatrix} \xi_- \\ \xi_- \end{pmatrix} \right\} \ . \tag{114}$$

Now when we take the expectation value, the cross-terms from the multiplication cancel exactly:

$$\langle \psi'_+ | S^3 | \psi'_+ \rangle = \frac{m}{4E} \left\{ \cosh^2(\eta/2)(2\xi_+^\dagger \xi_+) - \sinh^2(\eta/2)(2\xi_-^\dagger \xi_-) \right\} \ , \tag{115}$$

where we've introduced the factor of $2E$ for the relativstic normalization. The normalization of the spinors ξ is one, so

$$\langle \psi'_+ | S^3 | \psi'_+ \rangle = \frac{2m}{4m \cosh \eta} \left\{ \cosh^2(\eta/2) - \sinh^2(\eta/2) \right\} = \frac{1}{2 \cosh \eta} \ , \tag{116}$$

because after the boost by η, the particle has $E = m \cosh \eta$. In the ultrarelativistic limit $(\eta \to 0)$ this vanishes.

(c) Take the ultra-relativistic limit of the wavefunction. Show that the result is an eigenstate of the helicity operator, $\hat{p} \cdot \mathbf{S}$. Interpret the result of part (b) using this result.

The helicity operator is

$$\hat{p} \cdot \mathbf{S} = \hat{p} \cdot \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \hat{p} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \hat{p} \cdot \boldsymbol{\sigma} \end{pmatrix} \ , \tag{117}$$

where $\hat{p}$ is the particle's momentum. Since we have boosted in the x direction from rest:

$$\hat{p} = (1, 0, 0) \ , \tag{118}$$

so

$$h = \frac{1}{2} \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \ . \tag{119}$$

The boosted wavefunction

$$\psi'(x) = \sqrt{m} \left\{ \cosh(\eta/2) \begin{pmatrix} \xi_+ \\ \xi_+ \end{pmatrix} - \sinh(\eta/2) \begin{pmatrix} \xi_- \\ \xi_- \end{pmatrix} \right\} \ . \tag{120}$$

in the ultra-relativistic limit becomes

$$\psi'(x) = \sqrt{m} e^{\eta/2} \left\{ \begin{pmatrix} \xi_+ \\ \xi_+ \end{pmatrix} - \begin{pmatrix} \xi_- \\ \xi_- \end{pmatrix} \right\} = \sqrt{m} e^{\eta/2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \ , \tag{121}$$

because both hyperbolic functions diverge (the negative exponential term vanishes). We can act the helicity operator on the boosted wavefunction:

$$h\psi' = \frac{\sqrt{m}}{2} e^{\eta/2} \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \frac{\sqrt{m}}{2} e^{\eta/2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \tag{122}$$

$$= \frac{\sqrt{m}}{2} e^{\eta/2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = -\frac{1}{2} \sqrt{m} e^{\eta/2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = -\frac{1}{2} \psi' \ , \tag{123}$$

which is an eigenstate of helicity n with eigenvalue $-1/2$.

4 Schwartz 9.1.

Compton scattering in QED.

(a) Calculate the tree-level matrix elements for $\gamma\phi \to \gamma\phi$. Show that the Ward identity is satisfied.

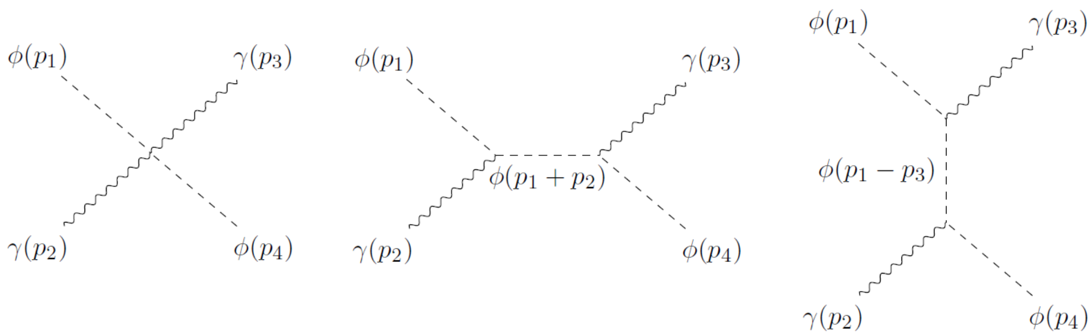

Figure 3: The tree-level Feynman diagrams describing the process $\gamma\phi \to \gamma\phi$. (Left) vertex only. (Center) s-channel. (Right) t-channel.

Let us label the process as follows:

$$\phi(p_1) + \gamma(p_2, \varepsilon_2) \to \gamma(p_3, \varepsilon_3^*) + \phi(p_4) \ , \tag{124}$$

where the ε are photon polarization vectors. The tree-level diagrams for this scattering process are shown in figure 3, the corresponding matrix elements will be $\mathcal{M}_1$, $\mathcal{M}_s$, and $\mathcal{M}_t$, respectively. The vertex-only diagram is trivial:

$$i\mathcal{M}_1 = 2ie^2 \varepsilon_2 \cdot \varepsilon_3^* \ . \tag{125}$$

Since the remaining two processes are mediated through a scalar propagator, we need the appropriate Feynman rule:

$$\frac{-i}{p^2 - m_\phi^2 + i\epsilon} \ . \tag{126}$$

Using the Feynman rule for the two scalar-one photon vertex yields, for the s-channel:

$$i\mathcal{M}_s = (-ie)^2 (p_1 + p)_\mu \varepsilon_2^\mu \left(\frac{-i}{p^2 - m_\phi^2} \right) (p_4 + p)_\nu \varepsilon_3^{*\nu} \ , \tag{127}$$

where the propagator momentum is $p = p_1 + p_2 = p_3 + p_4$. Inserting this yields

$$i\mathcal{M}_s = (-ie)^2 (2p_1 + p_2) \cdot \varepsilon_2 \left(\frac{-i}{(p_1 + p_2)^2 - m_\phi^2} \right) (2p_4 + p_3) \cdot \varepsilon_3^* \ , \tag{128}$$

but we have the conditions

$$\varepsilon_3^* \cdot p_3 = 0 = \varepsilon_2 \cdot p_2 \ , \tag{129}$$

so the matrix element reduces to

$$i\mathcal{M}_s = (-2ie)^2(p_1 \cdot \varepsilon_2)\left(\frac{-i}{(p_1+p_2)^2 - m_\phi^2}\right)(p_4 \cdot \varepsilon_3^*) \tag{130}$$

$$= 4ie^2\frac{(p_1 \cdot \varepsilon_2)(p_4 \cdot \varepsilon_3^*)}{s - m_\phi^2} \ . \tag{131}$$

The t-channel element is

$$i\mathcal{M}_t = (-ie)^2(p_1+p)_\mu \varepsilon_3^{*\mu}\left(\frac{-i}{p^2 - m_\phi^2}\right)(p_4+p)_\nu \varepsilon_2^\nu \ , \tag{132}$$

where the propagator momentum is $p = p_1 - p_3 = p_4 - p_2$. Inserting this, one obtains

$$i\mathcal{M}_t = (-ie)^2(2p_1 - p_3)\cdot\varepsilon_3^*\left(\frac{-i}{(p_1-p_3)^2 - m_\phi^2}\right)(2p_4 - p_2)\cdot\varepsilon_2 \ , \tag{133}$$

but again:

$$\varepsilon_2 \cdot p_2 = 0 = \varepsilon_3^* \cdot p_3 \ , \tag{134}$$

so

$$i\mathcal{M}_t = -(2ie)^2(p_1 \cdot \varepsilon_3^*)\left(\frac{-i}{(p_1-p_3)^2 - m_\phi^2}\right)(p_4 \cdot \varepsilon_2) \tag{135}$$

$$= 4ie^2\frac{(p_1 \cdot \varepsilon_3^*)(p_4 \cdot \varepsilon_2)}{t - m_\phi^2} \ . \tag{136}$$

Thus, the total matrix element is

$$\mathcal{M} = 2e^2\left\{\varepsilon_2 \cdot \varepsilon_3^* + 2\frac{(p_1 \cdot \varepsilon_2)(p_4 \cdot \varepsilon_3^*)}{s - m_\phi^2} + 2\frac{(p_1 \cdot \varepsilon_3^*)(p_4 \cdot \varepsilon_2)}{t - m_\phi^2}\right\} \ . \tag{137}$$

To check the Ward identity, we make the substitution $\varepsilon_3^* \to p_3$:

$$\mathcal{M} = 2e^2\left\{\varepsilon_2 \cdot p_3 + 2\frac{(p_1 \cdot \varepsilon_2)(p_4 \cdot p_3)}{s - m_\phi^2} + 2\frac{(p_1 \cdot p_3)(p_4 \cdot \varepsilon_2)}{t - m_\phi^2}\right\} \ . \tag{138}$$

Now, since the final states are on-shell, $p_1^2 = m_\phi^2 = p_4^2$ and $p_3^2 = 0 = p_2^2$, and so

$$t = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_\phi^2 - 2p_1 \cdot p_3 \tag{139}$$

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = p_3^2 + p_4^2 - 2p_3 \cdot p_4 = m_\phi^2 - 2p_3 \cdot p_4 \ , \tag{140}$$

which we can insert into the denominators to obtain

$$\mathcal{M} = 2e^2\left\{\varepsilon_2 \cdot p_3 + 2\frac{(p_1 \cdot \varepsilon_2)(p_4 \cdot p_3)}{-2p_3 \cdot p_4} + 2\frac{(p_1 \cdot p_3)(p_4 \cdot \varepsilon_2)}{-2p_1 \cdot p_3}\right\} \tag{141}$$

$$= 2e^2\left\{\varepsilon_2 \cdot p_3 - p_1 \cdot \varepsilon_2 - p_4 \cdot \varepsilon_2\right\} = 2e^2\epsilon_2 \cdot (p_3 - p_1 - p_4) \tag{142}$$

$$= 2e^2\epsilon_2 \cdot (p_4 - p_2 - p_4) = -2e^2 \cdot \epsilon_2 p_2 = 0 \ , \tag{143}$$

and so the Ward identity is satisfied.

(b) Calculate the cross section $\mathrm{d}\sigma/\mathrm{d}\cos\theta$ for this process as a function of the incoming and outgoing polarizations, $\varepsilon_\mu^{\text{in}}$ and $\varepsilon_\mu^{\text{out}}$, in the center-of-mass frame.

In terms of these polarization vectors, the matrix element is

$$\mathcal{M} = 2e^2 \left\{ \varepsilon^{\text{in}} \cdot \varepsilon^{\text{out}*} + 2\frac{(p_1 \cdot \varepsilon^{\text{in}})(p_4 \cdot \varepsilon^{\text{out}*})}{s - m_\phi^2} + 2\frac{(p_1 \cdot \varepsilon^{\text{out}*})(p_4 \cdot \varepsilon^{\text{in}})}{t - m_\phi^2} \right\} \tag{144}$$

$$= 2e^2 \left\{ \varepsilon^{\text{in}} \cdot \varepsilon^{\text{out}*} - \frac{(p_1 \cdot \varepsilon^{\text{in}})(p_4 \cdot \varepsilon^{\text{out}*})}{p_3 \cdot p_4} - \frac{(p_1 \cdot \varepsilon^{\text{out}*})(p_4 \cdot \varepsilon^{\text{in}})}{p_1 \cdot p_3} \right\} , \tag{145}$$

and we will use the definitions $\mathcal{M}_1, \mathcal{M}_s, \mathcal{M}_t$ respectively for these terms. If we define the initial photon momentum to be along the z axis, then the polarization vectors $\varepsilon_\mu^{\text{in/out}}$ must be perpendicular to the photon momentum. Since we have been working in the center-of-momentum frame we know the photon and scalar have anti-parallel momenta, and thus:

$$p_1 \cdot \varepsilon^{\text{in/out}(*)} = 0 , \tag{146}$$

and then $\mathcal{M}_s = \mathcal{M}_t = 0$. Finally, we obtain

$$\mathcal{M} = 2e^2 \varepsilon^{\text{in}} \cdot \varepsilon^{\text{out}*} , \tag{147}$$

so

$$|\mathcal{M}|^2 = 4e^4 \left(\varepsilon^{\text{in}} \cdot \varepsilon^{\text{out}*}\right) \left(\varepsilon^{\text{in}*} \cdot \varepsilon^{\text{out}}\right) . \tag{148}$$

The differential cross-section in the center-of-momentum frame is, by Schwartz equation 5.32

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 E_{cm}^2} \frac{p_f}{p_i} |\mathcal{M}|^2 , \tag{149}$$

where $p_i \equiv |\mathbf{p}_1| = |\mathbf{p}_2|$ (similarly with the final state) and $E_{cm} = E_1 + E_2 = E_3 + E_4$. Using this, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4e^4}{64\pi^2 E_{cm}^2} \frac{\sqrt{E_4^2 - m_\phi^2}}{\sqrt{E_1^2 - m_\phi^2}} \left(\varepsilon^{\text{in}} \cdot \varepsilon^{\text{out}*}\right) \left(\varepsilon^{\text{in}*} \cdot \varepsilon^{\text{out}}\right) , \tag{150}$$

but in the center of mass frame $E_1 = E_3$ by energy-momentum conservation. so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4}{4^2\pi^2 s} \left(\varepsilon^{\text{in}} \cdot \varepsilon^{\text{out}*}\right) \left(\varepsilon^{\text{in}*} \cdot \varepsilon^{\text{out}}\right) \tag{151}$$

$$= \frac{\alpha^2}{s} \left(\varepsilon^{\text{in}} \cdot \varepsilon^{\text{out}*}\right) \left(\varepsilon^{\text{in}*} \cdot \varepsilon^{\text{out}}\right) , \tag{152}$$

where $\alpha = e^2/4\pi$ and $s = E_{cm}^2$.

(c) Evaluate $\mathrm{d}\sigma/\mathrm{d}\cos\theta$ for $\varepsilon_\mu^{\text{in}}$ polarized in the plane of the scattering, for each $\varepsilon_\mu^{\text{out}}$.

We have enforced the polarizations are transverse to the z-axis, and can define them as

$$\varepsilon_1 \equiv \frac{1}{\sqrt{2}}(0, 1, i, 0) \quad \text{and} \quad \varepsilon_2 \equiv \frac{1}{\sqrt{2}}(0, 1, -i, 0) , \tag{153}$$

and $\varepsilon^{\text{in/out}}$ can take either value. These satisfy:

$$\varepsilon_1 \cdot \varepsilon_1 = -\frac{1}{2}(1^2 + i^2) = 0 \qquad \varepsilon_2 \cdot \varepsilon_1 = -\frac{1}{2}(1 - i^2) = -1 \tag{154}$$

$$\varepsilon_1 \cdot \varepsilon_2 = -\frac{1}{2}(1^2 - i^2) = -1 \qquad \varepsilon_2 \cdot \varepsilon_2 = -\frac{1}{2}(1 + (-i)^2) = 0 \tag{155}$$

$$\varepsilon_1 = \varepsilon_2^* \qquad \varepsilon_2 = \varepsilon_1^* \ , \tag{156}$$

the first four can be expressed $\epsilon_i \cdot \epsilon_j = \delta_{ij} - 1$. Consider a given $\varepsilon^{\text{out}} = \varepsilon_i$, then the cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{s}\left(\varepsilon^{\text{in}} \cdot \varepsilon_i^*\right)\left(\varepsilon^{\text{in}*} \cdot \varepsilon_i\right) = \frac{\alpha^2}{s}\left(\varepsilon^{\text{in}} \cdot \varepsilon_j\right)\left(\varepsilon^{\text{in}*} \cdot \varepsilon_i\right)(1 - \delta_{ij}) \ , \tag{157}$$

the only way this is nonzero is if $\varepsilon^{\text{in}} = \varepsilon_i$ as well:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{s}(\varepsilon_i \cdot \varepsilon_j)(\varepsilon_i^* \cdot \varepsilon_i)(1 - \delta_{ij}) = \frac{\alpha^2}{s}(\varepsilon_i \cdot \varepsilon_j)(\varepsilon_j \cdot \varepsilon_i)(1 - \delta_{ij}) \tag{158}$$

$$= \frac{\alpha^2}{s}(\varepsilon_i \cdot \varepsilon_j)^2 (1 - \delta_{ij}) \ . \tag{159}$$

Note this is to be expected for a process mediated by a spinless propagator. We can express the cross-section for each polarization as

$$\frac{\mathrm{d}\sigma(\epsilon_1 \to \epsilon_1)}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma(\epsilon_2 \to \epsilon_2)}{\mathrm{d}\Omega} = \frac{\alpha}{s} \tag{160}$$

$$\frac{\mathrm{d}\sigma(\epsilon_1 \to \epsilon_2)}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma(\epsilon_2 \to \epsilon_1)}{\mathrm{d}\Omega} = 0 \ . \tag{161}$$

Alternatively, we can use the linearly polarized basis:

$$\varepsilon_1 \equiv (0, 1, 0, 0) \quad \text{and} \quad \varepsilon_2 \equiv (0, 0, 1, 0) \ , \tag{162}$$

which satisfy $\epsilon_i \cdot \epsilon_j = (\delta_{ij} - 1)$, but $\epsilon_i^* = \epsilon_i$. Working in the center-of-momentum frame, we define the z axis to be along the incoming photon's momentum, which is transverse to these polarization states. The scatter occurs in the $y - z$ plane (*i.e.*, all x-components of momenta are zero), so polarization in the plane of scattering means

$$\varepsilon^{\text{in}} = \varepsilon_2 \ . \tag{163}$$

The cross-section is then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{s}\left(\varepsilon_2 \cdot \varepsilon^{\text{out}*}\right)\left(\varepsilon_2 \cdot \varepsilon^{\text{out}}\right) \ , \tag{164}$$

in the linearly polarized basis, the polarization basis vectors are real, so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{s}\left(\varepsilon_2 \cdot \varepsilon^{\text{out}}\right)^2 \ , \tag{165}$$

and due to the normalization condition, we get:

$$\frac{\mathrm{d}\sigma(\varepsilon_2 \to \varepsilon_1)}{\mathrm{d}\Omega} = 0 \tag{166}$$

$$\frac{\mathrm{d}\sigma(\varepsilon_2 \to \varepsilon_2)}{\mathrm{d}\Omega} = \frac{\alpha}{s} \ . \tag{167}$$

(d) Evaluate $\mathrm{d}\sigma/\mathrm{d}\cos\theta$ for $\varepsilon_\mu^{\mathrm{in}}$ polarized transverse to the plane of the scattering, for each $\varepsilon_\mu^{\mathrm{out}}$.

Again, we will work in the center-of-momentum frame with the linearly polarized basis. Here the incoming photon is polarized transverse to the plane of scattering $(y - z)$, so

$$\varepsilon^{\mathrm{in}} = \varepsilon_1 \ , \tag{168}$$

thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{s}\left(\varepsilon_1 \cdot \varepsilon^{\mathrm{out}}\right)^2 \ , \tag{169}$$

and again by the normalization of the polarization basis:

$$\frac{\mathrm{d}\sigma(\epsilon_1 \to \epsilon_1)}{\mathrm{d}\Omega} = \frac{\alpha^2}{s} \tag{170}$$

$$\frac{\mathrm{d}\sigma(\epsilon_1 \to \epsilon_2)}{\mathrm{d}\Omega} = 0 \ , \tag{171}$$

and again we see no polarization change.

(e) Show that when you sum (c) and (d) you get the same thing as having replaced $(\epsilon_\mu^{\mathrm{in}})^*\epsilon_\nu^{\mathrm{in}}$ with $-g_{\mu\nu}$ and $(\epsilon_\mu^{\mathrm{out}})^*\epsilon_\nu^{\mathrm{out}}$ with $-g_{\mu\nu}$.

We start from equation 152, using Lorentz indeces:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{s}(\varepsilon_\mu^{\mathrm{in}})(\varepsilon^{\mathrm{out}*})^\mu(\varepsilon^{\mathrm{in}*})_\nu(\varepsilon^{\mathrm{out}})^\nu = \frac{\alpha^2}{s}(\varepsilon^{\mathrm{out}*})^\mu(\varepsilon^{\mathrm{out}})^\nu(\varepsilon^{\mathrm{in}*})_\nu(\varepsilon^{\mathrm{in}})_\mu \tag{172}$$

$$= \frac{\alpha^2}{s}(-g^{\mu\nu})(-g_{\nu\mu}) = \frac{\alpha^2}{s}g^{\mu\nu}g_{\mu\nu} = 4\frac{\alpha^2}{s} \ . \tag{173}$$

If we sum the results for each combination of incoming and outgoing polarizations, we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma(\epsilon_1 \to \epsilon_1)}{\mathrm{d}\Omega} + \frac{\mathrm{d}\sigma(\epsilon_1 \to \epsilon_2)}{\mathrm{d}\Omega} + \frac{\mathrm{d}\sigma(\epsilon_2 \to \epsilon_1)}{\mathrm{d}\Omega} + \frac{\mathrm{d}\sigma(\epsilon_2 \to \epsilon_2)}{\mathrm{d}\Omega} = 2\frac{\alpha^2}{s} \ . \tag{174}$$

(f) Should this replacement work for any scattering calculation?

No, because it doesn't even work in this one...

5 Problem 2 - Matrix Element Recalulation.

The momenta definitions are

$$p_1^\mu = \frac{E_{cm}}{2}(1,0,0,\beta) \qquad p_3^\mu = \frac{E_{cm}}{2}(1,0,\beta\sin\theta,\beta\cos\theta) \tag{175}$$

$$p_2^\mu = \frac{E_{cm}}{2}(1,0,0,-\beta) \qquad p_4^\mu = \frac{E_{cm}}{2}(1,0,-\beta\sin\theta,-\beta\cos\theta)\ , \tag{176}$$

so

$$p_1 + p_2 = \frac{E_{cm}}{2}(2,0,0,0) \tag{177}$$

$$p_1 + p_3 = \frac{E_{cm}}{2}(2,0,\beta\sin\theta,\beta(1+\cos\theta)) \tag{178}$$

$$p_1 + p_4 = \frac{E_{cm}}{2}(2,0,-\beta\sin\theta,\beta(1-\cos\theta)) \tag{179}$$

$$p_2 + p_3 = \frac{E_{cm}}{2}(2,0,\beta\sin\theta,-\beta(1+\cos\theta)) \tag{180}$$

$$p_2 + p_4 = \frac{E_{cm}}{2}(2,0,-\beta\sin\theta,-\beta(1+\cos\theta)) \tag{181}$$

$$p_3 + p_4 = \frac{E_{cm}}{2}(2,0,0,0) \tag{182}$$

$$\tag{183}$$

and

$$p_3 - p_1 = \frac{E_{cm}}{2}(0,0,\beta\sin\theta,-\beta(1-\cos\theta)) \tag{184}$$

$$p_4 - p_1 = \frac{E_{cm}}{2}(0,0,-\beta\sin\theta,-\beta(1+\cos\theta)) \tag{185}$$

$$p_2 - p_1 = \frac{E_{cm}}{2}(0,0,0,-2\beta) \tag{186}$$

$$p_3 - p_2 = \frac{E_{cm}}{2}(1,0,\beta\sin\theta,\beta(1+\cos\theta)) \tag{187}$$

$$p_4 - p_2 = \frac{E_{cm}}{2}(1,0,-\beta\sin\theta,\beta(1-\cos\theta)) \tag{188}$$

$$p_4 - p_3 = \frac{E_{cm}}{2}(0,0,-2\beta\sin\theta,-2\beta\cos\theta)\ . \tag{189}$$

From the Lorentz condition we have

$$m_{\tilde{e}}^2 = p_1^2 = \frac{E_{cm}^2}{4}(1-\beta^2) \quad \Rightarrow \quad \beta = \pm\sqrt{1-\frac{4m_{\tilde{e}}^2}{E_{cm}^2}}\ , \tag{190}$$

with $|\beta| < 1$, and $E_{cm} = 2E_i$.

5.1 $\tilde{e}^-(p_1)\ \tilde{e}^-(p_2) \to \tilde{e}^-(p_3)\ \tilde{e}^-(p_4)$

5.1.1 t-channel.

$$i\mathcal{M}_t = \frac{ie^2}{(p_3-p_1)^2}(p_1+p_3)_\mu(p_2+p_4)^\mu \tag{191}$$

$$\begin{aligned}
(p_1+p_3)_\mu(p_2+p_4)^\mu &= \frac{E_{cm}}{2}(2,0,\beta\sin\theta,\beta(1+\cos\theta))\cdot\frac{E_{cm}}{2}(2,0,-\beta\sin\theta,-\beta(1+\cos\theta)) && (192)\\
&= \frac{E_{cm}^2}{4}\left(4-\left\{-\beta^2\sin^2\theta-\beta^2(1+\cos\theta)^2\right\}\right) && (193)\\
&= \frac{E_{cm}^2}{4}\left(4+\beta^2\sin^2\theta+\beta^2(1+\cos^2\theta+2\cos\theta)\right) && (194)\\
&= \frac{E_{cm}^2}{4}\left(4+\beta^2\sin^2\theta+\beta^2+\beta^2\cos^2\theta+2\beta^2\cos\theta\right) && (195)\\
&= \frac{E_{cm}^2}{2}\left(2+\beta^2(1+\cos\theta)\right) && (196)
\end{aligned}$$

$$\begin{aligned}
(p_3-p_1)^2 &= -\frac{E_{cm}^2}{4}\left(\beta^2\sin^2\theta+\beta^2(1-\cos\theta)^2\right) && (197)\\
&= -\frac{E_{cm}^2}{4}\left(\beta^2\sin^2\theta+\beta^2(1+\cos^2\theta-2\cos\theta)\right) && (198)\\
&= -\frac{E_{cm}^2}{4}\left(\beta^2\sin^2\theta+\beta^2+\beta^2\cos^2\theta-2\beta^2\cos\theta\right) && (199)\\
&= -\frac{E_{cm}^2}{4}\left(2\beta^2-2\beta^2\cos\theta\right) = -\frac{E_{cm}^2}{2}\beta^2\left(1-\cos\theta\right) && (200)
\end{aligned}$$

$$i\mathcal{M}_t = -(ie^2)\frac{2+\beta^2(1+\cos\theta)}{\beta^2\left(1-\cos\theta\right)} \tag{201}$$

5.1.2 u-channel.

$$i\mathcal{M}_u = \frac{ie^2}{(p_1 - p_4)^2}(p_1 + p_4)_\mu (p_2 + p_3)^\mu \tag{202}$$

$$\begin{aligned}
(p_1 + p_4)_\mu (p_2 + p_3)^\mu &= \frac{E_{cm}}{2}(2, 0, -\beta \sin\theta, \beta(1 - \cos\theta)) \cdot \frac{E_{cm}}{2}(2, 0, \beta \sin\theta, -\beta(1 + \cos\theta)) &(203)\\
&= \frac{E_{cm}^2}{4}\left(4 - \left\{-\beta^2 \sin^2\theta - \beta^2(1 - \cos\theta)(1 + \cos\theta)\right\}\right) &(204)\\
&= \frac{E_{cm}^2}{4}\left(4 - \left\{-\beta^2 \sin^2\theta - \beta^2(1 - \cos^2\theta)\right\}\right) &(205)\\
&= \frac{E_{cm}^2}{4}\left(4 - \left\{-\beta^2 \sin^2\theta - \beta^2 + \beta^2\cos^2\theta)\right\}\right) &(206)\\
&= \frac{E_{cm}^2}{4}\left(4 + \beta^2\left\{\sin^2\theta + 1 - \cos^2\theta)\right\}\right) = \frac{E_{cm}^2}{4}\left(4 + 2\beta^2 \sin^2\theta\right) &(207)\\
&= \frac{E_{cm}^2}{2}\left(2 + \beta^2 \sin^2\theta\right) &(208)
\end{aligned}$$

$$\begin{aligned}
(p_1 - p_4)^2 &= \frac{E_{cm}^2}{4}(0, 0, \beta \sin\theta, \beta(1 + \cos\theta))^2 = -\frac{E_{cm}^2}{4}\left\{\beta^2 \sin^2\theta + \beta^2(1 + \cos\theta)^2\right\} &(209)\\
&= -\frac{E_{cm}^2}{4}\left\{\beta^2 \sin^2\theta + \beta^2(1 + \cos^2\theta + 2\cos\theta)\right\} &(210)\\
&= -\frac{E_{cm}^2}{4}\left\{\beta^2 \sin^2\theta + \beta^2 + \beta^2\cos^2\theta + 2\beta^2\cos\theta)\right\} &(211)\\
&= -\frac{E_{cm}^2}{4}\left\{2\beta^2 + 2\beta^2\cos\theta)\right\} = -\frac{E_{cm}^2}{2}\beta^2\left\{1 + \cos\theta)\right\} &(212)
\end{aligned}$$

$$i\mathcal{M}_u = -ie^2\frac{2 + \beta^2 \sin^2\theta}{\beta^2(1 + \cos\theta)} \tag{213}$$

5.2 $\tilde{e}^-(p_1)\ \tilde{e}^+(p_2) \to \tilde{e}^-(p_3)\ \tilde{e}^+(p_4)$

5.2.1 s-channel.

$$i\mathcal{M}_s = \frac{ie^2}{(p_1+p_2)^2}(p_1-p_2)_\mu(p_3-p_4)^\mu \tag{214}$$

$$\begin{aligned}(p_1+p_4)_\mu(p_2+p_3)^\mu &= \frac{E_{cm}}{2}(2,0,-\beta\sin\theta,\beta(1-\cos\theta))\cdot\frac{E_{cm}}{2}(2,0,\beta\sin\theta,-\beta(1+\cos\theta)) && (215)\\ &= \frac{E_{cm}^2}{4}\left(4-\left\{-\beta\sin^2\theta-\beta^2(1-\cos\theta)(1+\cos\theta)\right\}\right) && (216)\\ &= (p_1+p_4)_\mu(p_2+p_3)^\mu = \frac{E_{cm}^2}{2}\left(2+\beta^2\sin^2\theta\right) && (217)\end{aligned}$$

$$(p_1+p_2)^2 = \frac{E_{cm}^2}{2}(2,0,0,0)\cdot\frac{E_{cm}}{2}(2,0,0,0) = E_{cm}^2 \tag{218}$$

$$i\mathcal{M}_s = ie^2\frac{2+\beta^2\sin^2\theta}{2} \tag{219}$$

5.2.2 t-channel.

$$i\mathcal{M}_t = -\frac{ie^2}{(p_1-p_3)^2}(p_1+p_3)_\mu(p_2+p_4)^\mu \tag{220}$$

$$\begin{aligned}(p_1+p_3)_\mu(p_2+p_4)^\mu &= \frac{E_{cm}}{2}(2,0,\beta\sin\theta,\beta(1+\cos\theta))\cdot\frac{E_{cm}}{2}(2,0,-\beta\sin\theta,-\beta(1+\cos\theta)) && (221)\\ &= \frac{E_{cm}^2}{4}\left(4-\left\{-\beta^2\sin^2\theta-\beta^2(1+\cos\theta)^2\right\}\right) && (222)\\ &= (p_1+p_3)_\mu(p_2+p_4)^\mu = \frac{E_{cm}^2}{2}\left(2+\beta^2(1+\cos\theta)\right) && (223)\end{aligned}$$

$$(p_1-p_3)^2 = (p_3-p_1)^2 = -\frac{E_{cm}^2}{2}\beta^2\left(1-\cos\theta\right) \tag{224}$$

$$i\mathcal{M}_t = ie^2\frac{2+\beta^2(1+\cos\theta)}{\beta^2\left(1-\cos\theta\right)} \tag{225}$$