

DYLAN J. TEMPLES: SOLUTION SET ONE

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1 Problem #1: Hydrogenic Atoms.

The Schrödinger equation for a Hydrogenic atom of nuclear charge Ze is

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{Ze^2}{r}\psi = \varepsilon\psi \quad (1)$$

where e , m , and \hbar are the electron charge, electron mass and Planck's constant over 2π , respectively. The radial distance of the electron relative to the nucleus is r and $\psi(\mathbf{r})$ is the probability amplitude for the electron to be measured at radial position \mathbf{r} .

1.1 Dimensionless Schrödinger Equation.

First, we begin by noting some scales set by the parameters and constants of the problem. The dimensions of each constant are

$$[[\hbar]] = E T = M L^2 T^{-1} \quad (2)$$

$$[[m]] = M \quad (3)$$

$$[[e]] = L^{3/2} M^{1/2} T^{-1} \quad (4)$$

$$[[\varepsilon]] = ML^2, \quad (5)$$

to make Equation 1 dimensionless first we must change our position coordinate r with dimensions of length to a dimensionless coordinate ρ . Using dimensional analysis, a conversion factor $1/a_Z$ can be formed from the parameters above with dimensions of inverse length:

$$\begin{cases} [[Ze^2]] &= \frac{L^3 M}{T^2} \\ [[\hbar^{-2}]] &= \frac{T^2}{L^4 M^2} \end{cases} \Rightarrow \left[\left[\frac{mZe^2}{\hbar^2} \right] \right] = L^{-1}, \quad (6)$$

we can now define the dimensionless coordinate

$$\rho = \frac{1}{a_Z} r = \frac{\hbar^2}{mZe^2} r. \quad (7)$$

Furthermore, we can define a dimensionless energy using the conversion factor with units of inverse energy $1/E_0$,

$$E = \frac{1}{E_Z} \varepsilon = \frac{\hbar^2}{m(Ze^2)^2} \varepsilon. \quad (8)$$

The Hydrogenic atom potential is entirely radial so the wave function is separable, and we can write in spherical coordinates

$$\psi(r, \theta, \phi) = R(r)Y(\theta, \phi) \equiv \frac{U(r)}{r} Y(\theta, \phi), \quad (9)$$

we can now use the radial equation given by Shankar page 353 with the potential in Equation 1 to write

$$-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} U(r) + \left(\frac{Ze^2}{r} + \frac{l(l+1)\hbar^2}{2mr^2} \right) U(r) = \varepsilon U(r), \quad (10)$$

which by adding in the conversion factors from above, we get

$$\varepsilon U(r) = -\frac{\hbar^2}{2m} \frac{a_Z^2}{a_Z^2} \frac{d^2}{dr^2} U(r) + \left(\frac{a_Z}{a_Z} \frac{Ze^2}{r} + \frac{a_Z^2}{a_Z^2} \frac{l(l+1)\hbar^2}{2mr^2} \right) U(r) \quad (11)$$

$$\varepsilon U(r) = -\frac{\hbar^2}{2m} \frac{1}{a_Z^2} \frac{d^2}{d\rho^2} U(r) + \left(\frac{Ze^2}{a_Z\rho} + \frac{l(l+1)\hbar^2}{2ma_Z^2\rho^2} \right) U(r) \quad (12)$$

$$\varepsilon U(\rho) = -\frac{\hbar^2}{2m} \left(\frac{mZe^2}{\hbar^2} \right)^2 \frac{d^2}{d\rho^2} U(\rho) + \left(\frac{mZe^2}{\hbar^2} \frac{Ze^2}{\rho} + \left(\frac{mZe^2}{\hbar^2} \right)^2 \frac{l(l+1)\hbar^2}{2m\rho^2} \right) U(\rho) \quad (13)$$

$$\varepsilon U(\rho) = -\frac{mZ^2e^4}{2\hbar^2} \frac{d^2}{d\rho^2} U(\rho) + \left(\frac{mZ^2e^4}{\hbar^2\rho} + \frac{mZ^2e^4}{\hbar^2} \frac{l(l+1)}{2\rho^2} \right) U(\rho) \quad (14)$$

$$\frac{\hbar^2}{mZ^2e^4} \varepsilon U(\rho) = -\frac{1}{2} \frac{d^2}{d\rho^2} U(\rho) + \left(\frac{1}{\rho} + \frac{l(l+1)}{2\rho^2} \right) U(\rho) \quad (15)$$

$$EU(\rho) = -\frac{1}{2} \frac{d^2U(\rho)}{d\rho^2} + \left(\frac{1}{\rho} + \frac{l(l+1)}{2\rho^2} \right) U(\rho) , \quad (16)$$

which is dimensionless. For the following sections we will be interested in the ground state only, which we will assume an isotropic probability distribution. Therefore the angular momentum quantum number l is zero so the third term on the right hand side of the above equation drops out.

1.2 Ground State Binding Energy.

Simply using dimensional analysis to form a constant with the dimensions of energy, we can estimate the binding energy of the electron. We have already done this by defining the energy scale E_z in section 1.1, so

$$E_z = \frac{m(Ze^2)^2}{\hbar^2} , \quad (17)$$

is the estimated electron binding energy.

1.3 Most Probable Position.

Given the radial wave function is $R = rU$, we get that

$$R(r) = \sqrt{\frac{2}{a_Z}} e^{-r/a_Z} , \quad (18)$$

using the definition of ρ . Then we can say the most probable radial position is the location of the maximum of the probability density distribution. This distribution is the integral of the radial wave function (because the distribution is isotropic) over the surface of a sphere of radius r

$$\int |R(\mathbf{r})|^2 r^2 dr d\Omega \propto r^2 e^{-2r/a_Z} , \quad (19)$$

where factors do not matter, we are only interested in the functional form. If we maximize this, and solve for the position coordinate we get

$$\frac{d}{dr} \left[r^2 e^{-2r/a_Z} \right] = 0 = \frac{2re^{-\frac{2r}{a_Z}} (a_Z - r)}{a_Z} \Rightarrow r = a_Z , \quad (20)$$

so the characteristic length scale identified in section 1.1 is the most probable radial location. This is the equivalent of the Bohr radius for the Hydrogenic atom with nuclear charge Ze .

1.4 Velocity Dispersion.

Using dimensional analysis again, we can construct a constant with units of length per time to serve as a characteristic velocity, as follows

$$\left[\left[\frac{e^2}{\hbar} \right] \right] = \frac{L}{T}, \quad (21)$$

so the characteristic velocity for the electron in the Hydrogenic atom potential is

$$\Delta v = \frac{Ze^2}{\hbar}. \quad (22)$$

This theory would fail if the electron velocities approach the speed of light, $\Delta v = c$, which happens when the atomic number Z is of the order $\hbar c/e^2$, about $Z = 137$.

2 Problem #2: Electron Bound to Dielectric Plane.

The force between an electron (mass m and charge e) and a dielectric film is given by a potential, $V(z) = -\alpha e^2/z$, for $z > 0$, where $0 < \alpha < 1$ is related to the dielectric constant of the film and the substrate on which the film resides (see Figure 1). In this example assume $\alpha = 0.01$. For $z \leq 0$ assume the potential is infinitely repulsive, thus preventing penetration of the electron into the dielectric film.

2.1 The Hamiltonian.

The coordinate basis representation of the Hamiltonian in the half space $z > 0$ is

$$H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{\alpha e^2}{z}, \quad (23)$$

using Cartesian coordinates, with the Laplacian represented as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (24)$$

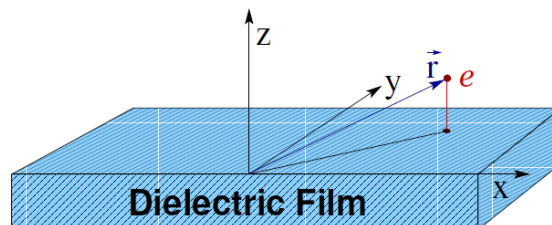


Figure 1: Diagram showing the coordinate system and dielectric film for problem #1.

2.2 Conservation Laws.

The operators of observable conserved quantities, or constants of motion, commute with the Hamiltonian. Immediately we can see that the Hamiltonian has no explicit time dependence (and commutes with itself) the conservation of energy is obeyed by this system. Additionally, the commutator of the Hamiltonian and one component of momentum is

$$[H, P_i] = a[|\mathbf{p}|^2, P_i] + b[z^{-1}, P_i], \quad (25)$$

which, because the total momentum operator commutes with any component, is zero except for $i = z$. In this case the z position would not commute with the z component of momentum. Therefore

$$[H, P_x] = 0 \quad (26)$$

$$[H, P_y] = 0, \quad (27)$$

so this system obeys conservation of momentum in the x and y directions.

2.3 Stationary States.

We can use separation of variables to write $\psi(x, y, z) = \chi(x, y)\phi(z)$, because the potential is solely dependent on the z coordinate. Therefore in x and y the electron is a free particle, so it obeys the Shrödinger equation given by

$$H\chi = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \chi = (E_x + E_y)\chi, \quad (28)$$

which can be solved by separating variables again and getting two exactly similar equations, one for x and y . The above equation is satisfied by two dimensional plane waves

$$\chi(x, y) = Ae^{i(\kappa_x x + \kappa_y y)} + Be^{-i(\kappa_x x + \kappa_y y)}, \quad (29)$$

after making the substitution $\kappa_i = (2mE_i/\hbar^2)^{1/2}$. The quantum numbers that describe their motion are the wave numbers κ_x and κ_y , which are free parameters, because the electron is a free particle.

We had used separation of variables to write Equation 28, this also yielded an equation for the amplitude for the electron to be a distance z above the film,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial z^2} - \frac{\alpha e^2}{z} \phi = E_z \phi, \quad (30)$$

2.4 Ground State Wave Function.

The $1/r$ form of the potential implies the electron's amplitude above the film must obey certain boundary conditions. The potential barrier at the film is infinite so $\phi(z=0) = 0$, and to be able to normalize the wave function $\phi(z \rightarrow \infty) = 0$. The simplest function that obeys these conditions is of the form

$$\phi(z) = Aze^{-az}, \quad (31)$$

it is easy to see this obeys the first condition. It obeys the second because the decaying exponential term dominates the linear term as $z \rightarrow \infty$. This ansatz can be substituted into Equation 30 to yield

$$-\frac{\hbar^2}{2m} \left[aAze^{-az} \left(a - \frac{2}{z} \right) \right] - \frac{\alpha e^2}{z} [Aze^{-az}] = E_z [Aze^{-az}] \quad (32)$$

$$-\frac{\hbar^2}{2m} \left[a \left(a - \frac{2}{z} \right) \right] - \frac{\alpha e^2}{z} = E_z, \quad (33)$$

we can now equate like powers of z term by term to get

$$\begin{cases} E_z &= -\frac{\hbar^2 a^2}{2m} \\ 0 &= \frac{\hbar^2 a}{m} - \alpha e^2 \end{cases} \Rightarrow \begin{cases} a &= \frac{\alpha e^2 m}{\hbar^2} \\ E_z &= \frac{\alpha^2 e^4 m}{2\hbar^2} \end{cases}. \quad (34)$$

With the value for a known, we can normalize the wave function

$$1 = \int_0^\infty |\phi(z)|^2 dz = \frac{A^2 \hbar^6}{4\alpha^3 e^6 m^3}, \quad (35)$$

so the wave function for the amplitude of the electron above the film is

$$\phi(z) = 2\sqrt{\frac{\alpha^3 e^6 m^3}{\hbar^6}} z e^{-\frac{\alpha e^2 m}{\hbar^2} z}, \quad (36)$$

which has probability density

$$P(z) = |\phi(z)|^2 = 4\frac{\alpha^3 e^6 m^3}{\hbar^6} z^2 \exp\left[-\frac{2\alpha e^2 m}{\hbar^2} z\right] \quad (37)$$

2.5 Most Probable Position.

To find the most probable position of the ground state electron above the film, we need to maximize the probability distribution of the electron's amplitude above the film. Setting the first derivative of the probability density equal to zero gives

$$0 = \frac{d}{dz} [A^2 z^2 e^{-az}]|_{z_0} = -2A^2 z_0 e^{-2az_0} (az_0 - 1) \Rightarrow z_0 = \frac{1}{a} = \frac{\hbar^2}{\alpha e^2 m}. \quad (38)$$

So the most probable height above the film in angstroms is

$$z_0 = \frac{(1.054\,56e-27 \text{ erg} \cdot \text{s})^2}{(0.01)(4.803e-10 \text{ esu})(9.1095e-28 \text{ g})} = 5.292\,04e-7 \text{ cm} = 5.29204 \text{ \AA}. \quad (39)$$

2.6 Ground State Binding Energy.

When separation of variables was done on the total wave function, the total energy was written as the sum of an energy for each coordinate. So the total energy is

$$E = E_z + E_x + E_y = \frac{\hbar^2}{2m}(\kappa_x^2 + \kappa_y^2) - \frac{\alpha^2 e^4 m}{2\hbar^2}, \quad (40)$$

however, the binding energy of the ground state is independent of the total energy (which is modified by the energy of the free particle energies in x and y). Therefore the binding energy is

$$E_0 = -\frac{(0.01)^2(4.803e-10 \text{ esu})^4(9.1095e-28 \text{ g})}{2(1.054\,56e-27 \text{ erg} \cdot \text{s})^2} = -2.179\,57e-15 \text{ erg} = -1.36e-3 \text{ eV}. \quad (41)$$

If a photon with an energy greater than the absolute value of this interacted with the electron, it would liberate it through photo-ionization. From the relationship between photon energy and wavelength we have

$$|E_0| = \frac{hc}{\lambda_{min}} \Rightarrow \lambda_{min} = \frac{(6.626e-27 \text{ erg} \cdot \text{s})(3e10 \text{ cm} \cdot \text{s}^{-1})}{2.179\,57e-15 \text{ erg}} = 0.0912013 \text{ cm}, \quad (42)$$

which is in the microwave region of the electromagnetic spectrum.

2.7 Energy Spectrum.

It is easy to see Equation 30 has the form of the Hydrogen atom with no angular momentum ($l = 0$), so we must have a spectrum of bound state energies and wave functions that are of the same form as the Hydrogen atom. In this case, the electron charge is associated with a factor of α , so by Shankar Equation 13.1.16, the energy spectrum is

$$E_n = -\frac{\alpha^2 e^4 m}{2\hbar^2 n^2} = \frac{E_0}{n^2}, \quad (43)$$

where the integer $n \geq 1$ is the number of the state. Note the ground state is $n = 1$ but has energy E_0 , this is just poor notation. The corresponding wave functions are the solutions to the Hydrogen atom's radial equation with $l = 0$, see Shankar Equations 12.1.22 and 13.1.23,

$$\phi_n(z) = \left(\frac{\alpha e^2 m}{\hbar^2 n}\right) z \exp\left[-\frac{\alpha e^2 m}{\hbar^2 n} z\right] L_{n-1}^1\left(\frac{\alpha e^2 m}{\hbar^2 n} z\right), \quad (44)$$

where $L_i^j(z)$ represent the associated Laguerre polynomials, which can be found in any standard reference.

3 Problem #3: Annular Potential.

The Hamiltonian for a non-relativistic electron moving in two dimensions in the presence of a potential $V(|\mathbf{r}|)$ is

$$H = \frac{1}{2m} |\mathbf{p}|^2 + V(\mathbf{r}) , \quad (45)$$

where

$$V(\mathbf{r}) = \begin{cases} V_0, & 0 \leq |\mathbf{r}| < R \\ 0, & R < |\mathbf{r}| < R + w \\ V_0, & R + w \leq |\mathbf{r}| \end{cases} , \quad (46)$$

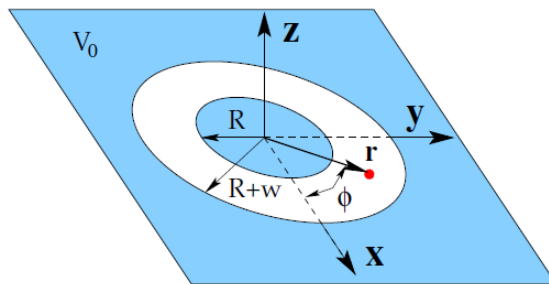


Figure 2: Diagram showing the coordinate system and toroidal barrier for problem #2.

is a toroidal barrier. For $V_0 \rightarrow \infty$ the electron is strictly confined within the strip $R < r < R + w$ as shown in Figure 2.

3.1 Angular Momentum.

The angular momentum of the system $L_z = xp_y - yp_x$ is a constant of the motion because it commutes with the Hamiltonian,

$$[H, L_z] \sim [|\mathbf{p}|^2, L_z] + [V_0, L_z] = [p_x^2 + p_y^2 + p_z^2, xp_y - yp_x] , \quad (47)$$

if we note that any component of momentum commutes with any component of position (as long as they're not the same component) and that the components of momentum commute with themselves, we can write

$$[H, L_z] \sim [p_x^2, xp_y] - [p_x^2, yp_x] + [p_y^2, xp_y] - [p_y^2, yp_x] + [p_z^2, xp_y] - [p_z^2, yp_x] = 0 , \quad (48)$$

so angular momentum is a conserved quantity.

The angular momentum projection operator for systems with rotational invariance (which this system exhibits) which is easy to remember (see Shankar page 335) is

$$L_z = -i\hbar \frac{\partial}{\partial \phi} , \quad (49)$$

note that the coordinate ϕ is the same in cylindrical and spherical coordinates. Additionally, we know the eigenvalues of this operator are

$$L_z |\Phi\rangle = \ell\hbar |\Phi\rangle , \quad (50)$$

where ℓ is the angular momentum z projection quantum number. If we write this as

$$-i\hbar \frac{\partial}{\partial \phi} \Phi(\phi) = \ell\hbar \Phi(\phi) , \quad (51)$$

we find the wave function for the azimuthal coordinate,

$$\Phi(\phi) = Ae^{i\ell\phi} , \quad (52)$$

where $\ell = 0, \pm 1, \pm 2 \dots$ and A is some normalization factor.

3.2 Stationary States.

We can write the Schrödinger equation in the coordinate basis with the Laplacian operator in cylindrical coordinates

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r, \phi, z) + V(r)\psi(r, \phi, z) = \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \psi(r, \phi, z) + V(r)\psi(r, \phi, z) = E\psi(r, \phi, z), \quad (53)$$

we can eliminate the z coordinate because the particle is constrained to move in the $x - y$ (r, ϕ) plane, so the derivative with respect to z is zero, and the third term in the Laplacian drops out. If we note the similarity between the second term in the Laplacian and the angular momentum operator, we can substitute

$$\frac{\partial}{\partial \phi} = \frac{i}{\hbar} L_z \quad \Rightarrow \quad \frac{\partial^2}{\partial \phi^2} = -\frac{1}{\hbar^2} L_z^2, \quad (54)$$

into the equation. Furthermore, we can separate variables and write $\psi(r, \phi) = S_n(r)\Phi_\ell(\phi)$, where n is the quantum number corresponding to radial motion. Consider now the limit in which $V_0 \rightarrow +\infty$, so the only region the electron can occupy is the toroid with no potential. This forces the radial wave function to be zero at the toroid boundaries

$$S_n(R) = S_n(R + w) = 0. \quad (55)$$

In this region Equation 53 is

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \frac{L_z^2}{\hbar^2} \right] S_n(r)\Phi_\ell(\phi) = -\frac{2m}{\hbar^2} E_{n\ell} S_n(r)\Phi_\ell(\phi), \quad (56)$$

if the operator on the left acts on the product of wave functions the first term will only effect the first wave function, likewise for the second. We can then replace the L_z^2 operator with it's eigenvalue and divide the entire equation by $\Phi_\ell(\phi)$,

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) - \frac{\ell^2}{r^2} \right] S_n(r) = -\kappa_{n\ell}^2 S_n(r), \quad (57)$$

where we've defined $\kappa_{n\ell} = (2mE_{n\ell}/\hbar^2)^{1/2}$. Now we can expand the derivatives in the Laplacian, move the energy term over, and finally multiply by r^2 to get

$$0 = r^2 \left[\frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \kappa_{n\ell}^2 - \frac{\ell^2}{r^2} \right] S_n(r) \quad (58)$$

$$= r^2 \frac{\partial^2}{\partial r^2} S_n(r) + r \frac{\partial}{\partial r} S_n(r) + (\kappa_{n\ell}^2 r^2 - \ell^2) S_n(r), \quad (59)$$

which is Bessel's equation (Arfken page 647), and has solutions of the form

$$S(r) = AJ_\ell(\kappa_{n\ell}r) + BY_\ell(\kappa_{n\ell}r), \quad (60)$$

where $J_i(x)$ and $Y_i(x)$ represent Bessel functions of the first and second (Neumann functions) kinds, respectively, and A and B are arbitrary constants. To satisfy the boundary conditions, we impose

$$0 = AJ_\ell(\kappa_{n\ell}R) + BY_\ell(\kappa_{n\ell}R) \quad (61)$$

$$0 = AJ_\ell(\kappa_{n\ell}(R + w)) + BY_\ell(\kappa_{n\ell}(R + w)). \quad (62)$$

We can combine these conditions to obtain a transcendental equation for the quantization conditions on n and ℓ ,

$$AJ_\ell(\kappa_{n\ell}R) + BY_\ell(\kappa_{n\ell}R) = AJ_\ell(\kappa_{n\ell}(R+w)) + BY_\ell(\kappa_{n\ell}(R+w)) = 0. \quad (63)$$

3.3 Lowest Energy Radial Modes.

Consider now, stationary states with the angular momentum eigenvalue $\ell = 0$. We can make the change of variables $x = r - R$, such that

$$dr = dx \quad (64)$$

$$r = x + R = R \left(1 + \frac{x}{R}\right) \simeq R, \quad (65)$$

because the x is on the domain $[0, w]$, if we make the approximation that $w \ll R$, then Equation 65 holds. If we substitute this variable into Equation 58 (without the factor of r^2 that was added) we get

$$0 = \left[\frac{\partial^2}{\partial x^2} + \frac{1}{R} \frac{\partial}{\partial x} + \kappa_{n0}^2 \right] S(x), \quad (66)$$

which is a linear, second-order ordinary differential equation with constant coefficients. We can solve the characteristic equation

$$\lambda^2 + \frac{1}{R}\lambda + \kappa_{n0}^2 = 0 \quad (67)$$

to find the two roots

$$\lambda = \frac{1}{2} \left(\pm \sqrt{\frac{1}{R^2} - 4\kappa_{n0}^2} - \frac{1}{R} \right) = -\frac{1}{2R} \pm \frac{1}{R} \sqrt{(-1)(4R^2\kappa_{n0}^2 - 1)}, \quad (68)$$

which gives the solution

$$S(x) = e^{-1/2R} \left(C_+ \exp \left[i \frac{1}{R} \sqrt{4R^2\kappa_{n0}^2 - 1} x \right] + C_- \exp \left[-i \frac{1}{R} \sqrt{4R^2\kappa_{n0}^2 - 1} x \right] \right), \quad (69)$$

where C_\pm is an arbitrary constant. We can throw out the positive exponential (first term) so the wave function is normalizable. We can also note the term in the radical is negative because $1/R^2$ is a small number for large R , so the complex exponential can be expressed as a linear combination of sines and cosines,

$$S(x) = e^{-1/2R} \left(\xi \cos \left[\frac{1}{R} \sqrt{4R^2\kappa_{n0}^2 - 1} x \right] + \zeta \sin \left[\frac{1}{R} \sqrt{4R^2\kappa_{n0}^2 - 1} x \right] \right), \quad (70)$$

where ξ and ζ are arbitrary constants. This equation must satisfy the boundary conditions $S(x=0) = S(x=w) = 0$. For the $s=0$ boundary, it must be that $\xi = 0$ so that the radial wave function is zero at the origin of the coordinate x . At the second boundary

$$0 = e^{-1/2R} \zeta \sin \left[\frac{1}{R} \sqrt{4R^2\kappa_{n0}^2 - 1} w \right] \Rightarrow \frac{1}{R} \sqrt{4R^2\kappa_{n0}^2 - 1} w = n\pi, \quad (71)$$

so the sine will always be zero here. This gives the quantization condition

$$\kappa_{n0}^2 = \frac{\left(\frac{n\pi R}{w}\right)^2 + 1}{4R^2} = \frac{2mE_{n0}}{\hbar^2}, \quad (72)$$

from the definition of $\kappa_{n\ell}$. So the energy spectrum is

$$E_{n0} = \frac{\hbar^2}{8mR^2} \left(\frac{n^2\pi^2 R^2}{w^2} + 1 \right). \quad (73)$$

The ground state is $n = 1$ and the first excited state is $n = 2$ so the energy gap is

$$\Delta E = E_{10} - E_{00} = \frac{\pi^2 \hbar^2}{8mw^2}. \quad (74)$$

Using the definition of the thermal energy, we can find the maximum temperature of a heat bath in thermal contact with the system, that keeps the electron in the ground state:

$$\Delta E = E_{therm} = kT_{max} = \frac{\pi^2 \hbar^2}{8mw^2}, \quad (75)$$

where k is Boltzmann's constant. Therefore the maximum temperature of the bath that keeps the electron in its lowest energy state is

$$T_{max} = \frac{1}{4} \frac{\pi^2 \hbar^2}{8kmw^2}. \quad (76)$$

3.4 Magnetic Field Perturbation.

Now we can consider the same geometry as before but in the presence of a magnetic field $\mathbf{B} = B\Theta(R - |\mathbf{r}|)\hat{\mathbf{z}}$ for all ϕ and z that penetrates the region, $r \leq R$. Note that $\Theta(x - x_0)$ is the Heaviside step function. Using Stokes' theorem we can write

$$\int_S d\mathbf{S} \cdot (\nabla \times \mathbf{A}) = \oint_{C(S)} d\boldsymbol{\ell} \cdot \mathbf{A}, \quad (77)$$

where the surface \mathbf{S} is the area of the circle of radius R in the $\hat{\mathbf{z}}$ direction, and the line element is $d\boldsymbol{\ell} = r d\phi \hat{\boldsymbol{\phi}}$. Since $\nabla \times \mathbf{A} = \mathbf{B}$, and if we take \mathbf{A} to only have an azimuthal component, we can write

$$\pi R^2 B = 2\pi r A = \Phi_B, \quad (78)$$

where Φ_B is the magnetic flux, in the region the Heaviside function is nonzero. Therefore the vector potential that preserves cylindrical symmetry is

$$\mathbf{A}(r) = \frac{R^2 B}{2r} \hat{\boldsymbol{\phi}}. \quad (79)$$

The Hamiltonian, in the region with no potential is

$$H = \frac{1}{2m} \left(|\mathbf{p}| - \frac{e}{c} \mathbf{A} \right)^2 = \frac{1}{2m} \left(|\mathbf{p}| - \frac{e R^2 B}{c 2r} \hat{\boldsymbol{\phi}} \right)^2. \quad (80)$$

Due to the Aharonov-Bohm Effect the wave function remains the same, but picks up a phase determined by the vector potential. The new wave function is then

$$\psi_B(\mathbf{r}) = \psi_0(\mathbf{r}) \exp \left[-\frac{ie}{\hbar c} \int \mathbf{A} \cdot d\boldsymbol{\ell} \right] = \psi_0(\mathbf{r}) \exp \left[-\frac{ie}{\hbar c} \int \mathbf{A} \cdot d\boldsymbol{\ell} \right] = \psi_0(\mathbf{r}) \exp \left[-\frac{ie}{\hbar c} \Phi_B \right], \quad (81)$$

where ψ_0 is the wave function with no magnetic field, $\psi_0 = S_n(r)\Phi_\ell(\phi)$. We can now write the angular momentum projection operator as

$$L_z \rightarrow L'_z = L_z - \frac{e}{c} \mathbf{A} \Rightarrow \frac{L_z^2}{r^2 \hbar^2} \rightarrow \frac{1}{r^2} \left[-i \frac{\partial}{\partial \phi} - \frac{e \Phi_B}{2\pi c \hbar} \right]^2, \quad (82)$$

when this acts on the azimuthal wave function it returns a new eigenvalue which can be substituted into the Schrödinger equation as in Equation 57, so we get the same equation but with the angular momentum projection quantum number replaced as

$$\ell^2 \rightarrow \left(\ell - \frac{e \Phi_B}{2\pi c \hbar} \right)^2, \quad (83)$$

so the total energy (found by repeating the calculation from section 3.3 with $\ell \neq 0$), Equation 73, is

$$E_{n\ell} = \frac{\hbar^2}{2m} \left[\frac{n^2 \pi^2}{4w^2} + \frac{4\ell^2 + 1}{4R^2} \right] \rightarrow E_{n\ell}^{(B)} = \frac{\hbar^2}{2m} \left[\frac{n^2 \pi^2}{4w^2} + \frac{4 \left(\ell - \frac{e \Phi_B}{2\pi c \hbar} \right)^2 + 1}{4R^2} \right]. \quad (84)$$